ARITHMETIC OF CHARACTERISTIC p SPECIAL L-VALUES (WITH AN APPENDIX BY V. BOSSER)

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Abstract. Recently the second author has associated a finite $\mathbf{F}_q[T]$-module H to the Carlitz module over a finite extension of $\mathbf{F}_q(T)$. This module is an analogue of the ideal class group of a number field.

In this paper we study the Galois module structure of this module H for 'cyclotomic' extensions of $\mathbf{F}_q(T)$. We obtain function field analogues of some classical results on cyclotomic number fields, such as the p-adic class number formula, and a theorem of Mazur and Wiles about the Fitting ideal of ideal class groups. We also relate the Galois module H to Anderson's module of circular units, and give a negative answer to Anderson's Kummer-Vandiver-type conjecture.

These results are based on a kind of equivariant class number formula which refines the second author's class number formula for the Carlitz module.

Contents

1. Introduction

1.1. Let q be a prime power and $A = \mathbf{F}_q[T]$ the polynomial ring in one variable T over a finite field $\mathbf{F}_q$ with q elements. Let $P \in A$ be monic and irreducible. The special L-values referred to in the title are values at $s = 1$ of ∞-adic and P-adic Goss L-functions associated with various characters of $(A/P)^\times$.

1.2. Let us first define the relevant ∞-adic L-values. Let $k_\infty = \mathbf{F}_q((T^{-1}))$ be the completion of k at the place at infinity. Let F be a field extension of $\mathbf{F}_q$ and let $\chi\colon (A/P)^\times \to F^\times$ be a homomorphism. We define:

$$L(1,\chi) := \sum_{a \in A_+} \frac{\chi(a)}{a} \in F \otimes_{\mathbf{F}_q} k_\infty, \tag{1}$$

where A_+ denotes the set of monic elements of A, and where for a divisible by P we define

$$\chi(a) := \begin{cases} 1 & \text{if } \chi = 1 \\ 0 & \text{if } \chi \neq 1 \end{cases}$$

This series converges.

1.3. For the P-adic L-values consider the completion $A_P = \varprojlim_n A/P^n$ and a homomorphism $\chi\colon (A/P)^\times \to A_P^\times$. Then we define

$$L_P(1,\chi) := \sum_{n\geq 0} \sum_{a \in A_{n,+}} \frac{\chi(a)}{a} \in A_P, \tag{2}$$

where $A_{n,+} \subset A$ is the set of monic elements of degree n, and where this time we put $\chi(a) = 0$ when P divides a, for all χ. Unlike the ∞-adic case (1), the convergence of the infinite sum (2) is not a priori obvious. However, it follows from either [6, Lemma 3.6.7] or [1, §4.10] that the infinite sum (2), with the terms grouped as indicated, converges in A_P.

1.4. In this paper we study arithmetic properties of these special L-values. In particular, we consider function field versions of various results about cyclotomic number fields such as the Kummer-Vandiver problem, the theorem of Mazur and Wiles relating the Fitting ideal of class groups to Bernoulli numbers, and the p-adic class number formula.

1.5. Although this may not be a priori clear, the arithmetic properties encoded by these L-values are closely related to the *Carlitz module* (a particular Drinfeld module), and to the "unit module" and "class module" associated to the Carlitz module by the second author [12, 13]. One of the principal objectives of this paper is to relate the Galois module structure of these modules to the above special L-values.

In the next section we recall some of the theory of the Carlitz module, and state our main results. Along the way we fix some notation.

2. Statement of the principal results

2.1. Let $A = \mathbf{F}_q[T]$. For any A-algebra R denote by $\mathrm{C}(R)$ the A-module whose underlying $\mathbf{F}_q$-vector space is R, equipped with the unique A-module structure

$$A \times \mathrm{C}(R) \to \mathrm{C}(R)$$

satisfying

$$(T, r) \mapsto Tr + r^q$$

for all $r \in R$. The resulting functor C from the category of A-algebras to the category of A-modules is called the *Carlitz module*. It is a *Drinfeld module* of rank 1. See [7] for more background on Drinfeld modules and on the Carlitz module.

2.2. Let $k = \mathbf{F}_q(T)$ be the fraction field of A. There is a unique power series $\exp_{\mathrm{C}} X$ of the form

$$\exp_{\mathrm{C}} X = X + e_1 X^q + e_2 X^{q^2} + \cdots \in k[[X]]$$

such that

$$\exp_{\mathrm{C}}(TX) = T \exp_{\mathrm{C}} X + (\exp_{\mathrm{C}} X)^q. \tag{3}$$

This power series is called the *Carlitz exponential*. If F is a finite extension of $k_\infty = \mathbf{F}_q((T^{-1}))$ then the power series $\exp_{\mathrm{C}}$ defines an entire function on F and the functional equation (3) implies that $\exp_{\mathrm{C}}$ defines an A-module homomorphism $\exp_{\mathrm{C}}\colon F \to \mathrm{C}(F)$.

2.3. Now let K be a finite extension of k. Let $\mathcal{O}_K$ be the integral closure of A in K. Define

$$K_\infty := K \otimes_k k_\infty = \prod_{v|\infty} K_v.$$

Consider the map of A-modules

$$\partial\colon \mathrm{C}(\mathcal{O}_K) \times K_\infty \to \mathrm{C}(K_\infty), \quad (x,\gamma) \mapsto x - \exp_{\mathrm{C}} \gamma.$$

It is shown in [12] that the A-module

$$\mathrm{U}(\mathcal{O}_K) := \ker \partial$$

is finitely generated, and that the A-module

$$\mathrm{H}(\mathcal{O}_K) := \operatorname{coker} \partial$$

is finite. We identify $\mathrm{U}(\mathcal{O}_K)$ with the submodule of K_∞ consisting of precisely those elements whose image under $\exp_{\mathrm{C}}$ is in $\mathrm{C}(\mathcal{O}_K)$.

$\mathrm{H}(\mathcal{O}_K)$ is an A-module analogue of the ideal class group of a number field and $\mathrm{U}(\mathcal{O}_K)$ is an A-module analogue of the lattice of logarithms of units in a number field. The exponential $\exp_{\mathrm{C}}$ restricts to a map $\mathrm{U}(\mathcal{O}_K) \to \mathrm{C}(\mathcal{O}_K)$. Unlike what happens for units in number fields, the cokernel of this map is not finite, in fact by [10] it is not even a finitely generated A-module.

2.4. Let $P \in A$ be monic irreducible and denote its degree by d. Let K be the splitting field of the P-torsion of the Carlitz module over k. In the rest of this paper K will denote this particular finite extension of k, associated to the fixed prime P.

K/k is an abelian extension of degree $q^d - 1$. Its Galois group Δ is canonically isomorphic with $(A/P)^\times$. The extension is unramified away from P and ∞.

2.5. Our first result is a kind of *equivariant class number formula*, relating the special values $L(1,\chi)$ to the $A[\Delta]$-modules $\mathrm{H}(\mathcal{O}_K)$ and $\mathrm{U}(\mathcal{O}_K)$.

To state the theorem, it is convenient to group all the $L(1,\chi)$ together in one equivariant L-value. Let F be an extension of $\mathbf{F}_q$. For a character $\chi\colon \Delta \to F^\times$ let $e_\chi \in F[\Delta]$ be the corresponding idempotent:

$$e_\chi := -\sum_{\sigma\in\Delta} \chi(\sigma)^{-1}\sigma.$$

Now assume that F contains a field of q^d elements, so that every F-linear representation of Δ is a direct sum of one-dimensional representations. Then we define

$$L(1,\Delta) := \sum_{\chi\colon \Delta\to F^\times} L(1,\chi)e_\chi \in F \otimes_{\mathbf{F}_q} k_\infty[\Delta]. \tag{4}$$

We have that $L(1,\Delta)$ lies in $k_\infty[\Delta]^\times$, and that it does not depend on F.

K_∞ is free of rank one as a $k_\infty[\Delta]$-module, and it contains sub-$A[\Delta]$-modules $\mathcal{O}_K$ and $\mathrm{U}(\mathcal{O}_K)$.

Theorem A. *$\mathcal{O}_K$ and $\mathrm{U}(\mathcal{O}_K)$ are free of rank one as $A[\Delta]$-modules and*

$$L(1,\Delta)\cdot\mathcal{O}_K = \mathrm{Fitt}_{A[\Delta]}\,\mathrm{H}(\mathcal{O}_K)\cdot\mathrm{U}(\mathcal{O}_K)$$

inside K_∞.

This is an equivariant refinement of (a special case of) the class number formula of [13], and our proof (see section 6) follows closely the argument of *loc. cit.*

2.6. For our further results we need to split $\mathrm{H}(\mathcal{O}_K)$ into an "odd" and an "even" part, which we now define. Note that we have $\mathbf{F}_q^\times \subset \Delta = (A/P)^\times$. Let M be an $A[\Delta]$-module. Then M decomposes as

$$M = \bigoplus_{\chi\colon \mathbf{F}_q^\times\to\mathbf{F}_q^\times} M(\chi)$$

where $\mathbf{F}_q^\times \subset \Delta$ acts on $M(\chi)$ through the character χ. Let $\omega\colon \mathbf{F}_q^\times \to \mathbf{F}_q^\times$ be the tautological character $x\mapsto x$. We define the *odd* part of M as

$$M^- = M(\omega)$$

and the *even* part of M as

$$M^+ = \bigoplus_{\chi\neq\omega} M(\chi).$$

Clearly we have $M = M^+\oplus M^-$ for every $A[\Delta]$-module M. Correspondingly the ring $A[\Delta]$ factors as $A[\Delta]^+\times A[\Delta]^-$.

The subgroup $\mathbf{F}_q^\times$ of Δ is the decomposition group at ∞ in K/k, and as such it is analogous to the subgroup generated by complex conjugation in Galois group of a cyclotomic extension of $\mathbf{Q}$. Our use of the terms "odd" and "even" is motivated by this analogy.

2.7. Similarly, if F is a field extension of $\mathbf{F}_q$ and $\chi\colon\Delta\to F^\times$ a homomorphism then we say that χ is *odd* if χ restricts to the identity map on $\mathbf{F}_q^\times\subset\Delta$, and *even* otherwise. If F contains a field of q^d elements and M is an $F[\Delta]$-module then we have

$$M = \bigoplus_{\chi\colon\Delta\to F^\times} e_\chi M$$

and M^+ and M^- are the submodules obtained by restricting the direct sum to even or odd χ respectively.

2.8. We now consider the odd part $\mathrm{H}(\mathcal{O}_K)^-$. We will give a formula for the Fitting ideal of the $A[\Delta]$-module $\mathrm{H}(\mathcal{O}_K)^-$ similar to the theorem of Mazur-Wiles [8, p. 216, Theorem 2] relating the p-part of the class group of $\mathbf{Q}(\zeta_p)$ to generalized Bernoulli numbers. However, we give a full description of the Fitting ideal, not only of its P-part.

In §5 we will see that $\mathcal{O}_K$ is free of rank one as an $A[\Delta]$-module. Let η be a generator of $\mathcal{O}_K$ as $A[\Delta]$-module and let $\lambda\in\mathcal{O}_K$ be a non-zero P-torsion element of $\mathrm{C}(\mathcal{O}_K)$. Let F be a field containing $\mathbf{F}_q$ and $\chi\colon\Delta\to F^\times$ a homomorphism. Then there is a unique $B_{1,\chi}\in F\otimes_{\mathbf{F}_q} k$ such that

$$e_\chi\lambda^{-1} = B_{1,\chi}e_\chi\eta$$

in $F\otimes_{\mathbf{F}_q} K$.

Theorem B. *Let F be a field containing $\mathbf{F}_q$ and let $\chi\colon \Delta \to F$ be an odd character. Consider the ideal $I = \operatorname{Fitt} e_\chi(F \otimes_{\mathbf{F}_q} \mathrm{H}(\mathcal{O}_K))$ in $F \otimes_{\mathbf{F}_q} A$. Then*

1. *$I = (1)$ if $\chi = 1$ (and then $q = 2$);*
2. *$I = ((T - \chi(T))B_{1,\chi^{-1}})$ if χ extends to a ring homomorphism $A/P \to F$;*
3. *$I = (B_{1,\chi^{-1}})$ otherwise.*

Note that $B_{1,\chi}$ depends on the choice of λ and η, but only up to a scalar in $F^\times$. In §5 we will single out for each χ a particular $B_{1,\chi}$, independent of choices. We will call these *generalized Bernoulli-Carlitz numbers.*

2.9. For all positive integers n we define $\mathrm{BC}'_n \in k$ by the power series identity

$$\frac{X}{\exp_{\mathrm{C}} X} = \sum_{n\geq 0} \mathrm{BC}'_n\, X^n.$$

These BC'_n are (up to a normalisation factor) the *Bernoulli-Carlitz numbers* introduced by Carlitz, who related them to certain Goss zeta values. In §8 we establish congruences relating the $B_{1,\chi}$ to Bernoulli-Carlitz numbers and use these to obtain a new proof of the *Herbrand-Ribet theorem* of [14]:

Theorem C. *Let $\omega\colon \Delta \to (A/P)^\times$ be the tautological character. Let $1 < n < q^d - 1$ be divisible by $q - 1$. Then*

$$e_{\omega^{1-n}}(A/P \otimes_A \mathrm{H}(\mathcal{O}_K)) \neq 0$$

if and only if $v_P(\mathrm{BC}'_n) > 0$.

2.10. We have no complete description of the Fitting ideal of the even part $\mathrm{H}(\mathcal{O}_K)^+$, but give a kind of P-adic class number formula involving the P-part of $\mathrm{H}(\mathcal{O}_K)^+$. To state this formula, we need to consider a P-adic version of the module $\mathcal{U}$.

Let $\mathcal{O}_{K,P}$ be the completion of $\mathcal{O}_K$ at the unique prime above $P \in A$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{K,P}$. Note that the subgroup $\mathfrak{m}$ of $\mathcal{O}_{K,P}$ is stable under the Carlitz A-action. We denote the resulting A-module by $\mathrm{C}(\mathfrak{m}) \subset \mathrm{C}(\mathcal{O}_{K,P})$. The A-action extends uniquely to a continuous A_P-module structure on $\mathrm{C}(\mathfrak{m})$. Now let $\mathcal{U}$ be the image of $\mathrm{U}(\mathcal{O}_K)$ in $\mathrm{C}(\mathcal{O}_K)$ and let $\bar{\mathcal{U}}$ be the topological closure of $\mathcal{U} \cap \mathrm{C}(\mathfrak{m})$ inside $\mathrm{C}(\mathfrak{m})$. Then $\bar{\mathcal{U}}$ is a sub-$A_P[\Delta]$-module of $\mathrm{C}(\mathfrak{m})$.

2.11. The residue field $A_P \to A/P$ has a canonical section, giving A_P the structure of an A/P-algebra. In particular, every $A_P[\Delta]$-module M decomposes as

$$M = \bigoplus_\chi e_\chi M$$

where χ runs over all homomorhpisms $\chi\colon \Delta \to A_P^\times$, and where $e_\chi \in A_P[\Delta]$ is the idempotent associated to χ. We call a homomorphism $\chi\colon \Delta \to A_P^\times$ *odd* if its restriction to $\mathbf{F}_q^\times$ is the inclusion map $\mathbf{F}_q^\times \subset A_P^\times$, and *even* otherwise.

Theorem D. *Let $\chi\colon \Delta \to A_P^\times$ be even. Then*

$$\operatorname{length}_{A_P} e_\chi\,(A_P \otimes_A \mathrm{H}(\mathcal{O}_K)) + \operatorname{length}_{A_P} e_\chi \frac{\mathrm{C}(\mathfrak{m})}{\bar{\mathcal{U}}} = v_P(L_P(1,\chi)).$$

It is not a priori clear that $e_\chi(\mathrm{C}(\mathfrak{m})/\bar{\mathcal{U}})$ is finite and that $L_P(1,\chi) \neq 0$, but using a P-adic Baker-Brumer theorem of Vincent Bosser (see the appendix) we show the following Leopoldt-type result:

Theorem E. *If $\chi\colon \Delta \to A_P^\times$ is even then $e_\chi \bar{\mathcal{U}} \neq 0$ and $L_P(1,\chi) \neq 0$.*

We also show that $L_P(1,\chi) = 0$ for odd χ.

2.12. An important ingredient in the proof of Theorem D is Anderon's module $\mathcal{L}$ of *special points* [1]. This is a finitely generated submodule of $\mathrm{C}(\mathcal{O}_K)$, constructed through explicit generators. It is a Carlitz module analogue of the group of circular units (also known as cyclotomic units) in cyclotomic number fields. We refer to section 7 for the definition.

Recall that $\mathcal{U}$ is the image of $\mathrm{U}(\mathcal{O}_K)$ in $\mathrm{C}(\mathcal{O}_K)$. In §7 we will show

Theorem F. *The divisible closure of $\mathcal{L}$ in $\mathrm{C}(\mathcal{O}_K)$ is $\mathcal{U}$, the quotient $\mathcal{U}/\mathcal{L}$ is finite and we have*

$$\mathrm{Fitt}_{A[\Delta]}\,\mathcal{U}/\mathcal{L} = \mathrm{Fitt}_{A[\Delta]}\,\mathrm{H}(\mathcal{O}_K)^+.$$

As in the classical case, we do not expect $\mathcal{U}/\mathcal{L}$ and $\mathrm{H}(\mathcal{O}_K)^+$ to be isomorphic $A[\Delta]$-modules in general.

2.13. Motivated by the *Kummer-Vandiver conjecture*, Anderson had conjectured [1, §4.12] that the P-torsion of $\mathcal{U}/\mathcal{L}$ is trivial, and we now see that this is equivalent with the statement that $\mathrm{H}(\mathcal{O}_K)^+$ has trivial P-torsion. Recently we have found examples where the the latter does not hold [3], and we therefore conclude that also Anderson's conjecture is false. For example:

Theorem G. *Let $q = 3$ and $P = T^9 - T^6 - T^4 - T^3 - T^2 + 1$ in $\mathbf{F}_3[T]$. Then $\mathcal{U}/\mathcal{L}$ has non-trivial P-torsion.*

3. $A[\Delta]$-modules

3.1. Let $P \in A$ be an irreducible element of degree d, and let $\Delta = (A/P)^\times$. In this section we collect some elementary facts on the structure of $A[\Delta]$-modules, and fix some notation.

Note that $\mathbf{F}_q[\Delta] = \prod_i F_i$ for some finite field extensions $F_i/\mathbf{F}_q$. As a consequence we have $A[\Delta] = \prod_i F_i[T]$. In particular $A[\Delta]$ is a principal ideal ring.

3.2. If M is a finite $A[\Delta]$-module then there are ideals $I_1, \ldots, I_n$ such that

$$M \cong A[\Delta]/I_1 \oplus \cdots \oplus A[\Delta]/I_n.$$

The *Fitting ideal* of M is the ideal

$$\mathrm{Fitt}_{A[\Delta]}\, M := I_1 \cdots I_n.$$

3.3. Every ideal I of finite index in $A[\Delta]$ has a unique normalized generator f such that for every i the component $f_i \in F_i[T]$ of f is monic. If M is a finite $A[\Delta]$-module then we denote by $[M]_{A[\Delta]}$ this normalized generator of $\mathrm{Fitt}_{A[\Delta]}\, M$.

3.4. Let F be an extension of $\mathbf{F}_q$ and $\chi\colon \Delta \to F^\times$ a homomorphism. Consider the element

$$e_\chi := -\sum_{\sigma\in\Delta} \chi^{-1}(\sigma)\sigma \in F[\Delta]$$

Then e_χ is an idempotent and $\sigma e_\chi = \chi(\sigma) e_\chi$ for all $\sigma \in \Delta$.

3.5. Let F be a field containing a field of q^d elements. Then the ring $F \otimes_{\mathbf{F}_q} A[\Delta]$ factors as

$$F \otimes_{\mathbf{F}_q} A[\Delta] = \prod_{\chi\colon \Delta \to F^\times} (F \otimes_{\mathbf{F}_q} A) e_\chi$$

where e_χ is the idempotent corresponding to the character χ. If M is an $A[\Delta]$-module, then we have a decomposition

$$F \otimes_{\mathbf{F}_q} M = \bigoplus_{\chi} e_\chi (F \otimes_{\mathbf{F}_q} M).$$

3.6. Let F be an extension of $\mathbf{F}_q$ containing a field of q^d elements and let $\mathrm{Frob}\colon F \to F$ be the q-Frobenius $x \mapsto x^q$. Then for an

$$\alpha = \sum \alpha(\chi) e_\chi \in F \otimes_{\mathbf{F}_q} A[\Delta]$$

with $\alpha(\chi) \in F \otimes_{\mathbf{F}_q} A$ for all χ we have that α lies in $A[\Delta]$ if and only if

$$\alpha(\chi^q) = (\mathrm{Frob} \otimes \mathrm{id})\alpha(\chi)$$

for all χ.

3.7. Let V be a $k_\infty[\Delta]$-module which is free of rank one. An $A[\Delta]$-*lattice* Λ in V is a sub $A[\Delta]$-module $\Lambda \subset V$, free of rank one. If Λ_1 and Λ_2 are $A[\Delta]$-lattices in V then there is an $f \in k_\infty[\Delta]$ so that $\Lambda_2 = f\Lambda_1$. Moreover, this f is unique if we normalize it analoguously to 3.3, by demanding that for every i its component $f_i \in F_i((T^{-1}))$ has leading coefficient 1. We denote this normalized f by $[\Lambda_1 : \Lambda_2]_{A[\Delta]}$.

4. Elementary properties of the cyclotomic function field K

In this section we collect some elementary facts about the field extension K/k and about the Carlitz module over K. We refer to [11, §12] and [7, §3] for the proofs.

4.1. Recall that K denotes the spliting field of the P-torsion of the Carlitz module over k, and $\Delta = \mathrm{Gal}(K/k)$. We have $\mathrm{C}[P](K) \cong A/P$ and the canonical map

$$\omega\colon \Delta \to \mathrm{Aut}_A\, \mathrm{C}[P](K) = (A/P)^\times$$

is an isomorphism, which we use to identify Δ with $(A/P)^\times$.

The field of constants of K is $\mathbf{F}_q$. The extension K/k is unramified away from P and ∞. For a monic irreducible $f \in A$ which is coprime with P we have that $\omega(\mathrm{Frob}_{(f)}) = \bar{f} \in (A/P)^\times$. The prime P is totally ramified in K/k.

4.2. Let $\lambda \in K$ be a generator of $\mathrm{C}(K)[P]$. Then λ is integral over A, so $\lambda \in \mathcal{O}_K$. We have $\mathcal{O}_K = A[\lambda]$. Moreover, λ is a generator of the unique prime ideal of $\mathcal{O}_K$ that lies above (P).

4.3. Let k_∞^a be an algebraic closure of k_∞. Then the exponential map defines a short exact sequence

$$0 \longrightarrow A\bar{\pi} \longrightarrow k_\infty^a \stackrel{\exp_{\mathrm{C}}}{\longrightarrow} \mathrm{C}(k_\infty^a) \longrightarrow 0$$

with

$$\bar{\pi} = \left(\sqrt[q-1]{-T}\right)^q \prod_{n=1}^{\infty} \left(1 - T^{1-q^n}\right)^{-1} \in k_\infty(\sqrt[q-1]{-T})$$

for any choice of $(q-1)$-st root of $-T$. The field $k_\infty(\bar{\pi})$ has degree $q-1$ over k_∞.

4.4. The element

$$\lambda = \exp_{\mathrm{C}}(\bar{\pi}/P) \in k_\infty(\bar{\pi})$$

is a generator of $\mathrm{C}[P](k_\infty^a)$. Since $K = k(\lambda)$ has degree $q^d - 1$ over k, we find that there are $(q^d-1)/(q-1)$ places above ∞ in K and for each such place v we have

$$K_v \cong k_\infty(\lambda) = k_\infty(\bar{\pi}).$$

4.5. Let Λ be the kernel of $\exp_{\mathrm{C}}\colon K_\infty \to \mathrm{C}(K_\infty)$. Then we have a short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \mathrm{U}(\mathcal{O}_K) \stackrel{\exp_{\mathrm{C}}}{\longrightarrow} \mathcal{U} \longrightarrow 0.$$

By 4.3 and 4.4 we have that the A-module Λ is free of rank $(q^d-1)/(q-1)$ and $\mathrm{U}(\mathcal{O}_K)$ is free of rank $q^d - 1$.

4.6. The Galois group of the Kummer extension $k_\infty(\bar{\pi})/k_\infty$ is naturally isomorphic to $\mathbf{F}_q^\times$, and acts on $\lambda = \exp_{\mathrm{C}}(\bar{\pi}/P) \in k_\infty(\bar{\pi})$ via the tautological character $\mathrm{id}\colon \mathbf{F}_q^\times \to \mathbf{F}_q^\times$. We conclude that the subgroup $\mathbf{F}_q^\times \subset \Delta$ is both the inertia group and decomposition group at ∞. We also see that $\Lambda = \Lambda^-$.

4.7. Let $Q \in A$ be the largest multiple of P so that $(A/Q)^\times = (A/P)^\times$. Then we have

$$\mathrm{C}(\mathcal{O}_K)_{\mathrm{tors}} = \mathrm{C}(K)_{\mathrm{tors}} = \mathrm{C}(K)[Q] \cong A/Q$$

and

$$\mathrm{C}(\mathcal{O}_K)^-_{\mathrm{tors}} = \mathrm{C}(\mathcal{O}_K)_{\mathrm{tors}}.$$

An easy computation shows that $Q = P$ if $q > 2$ and that Q is the least common multiple of P and $T(T+1)$ if $q = 2$.

5. Gauss-Thakur sums and generalized Bernoulli-Carlitz numbers

5.1. Fix a generator $\lambda \in K$ of the P-torsion of the Carlitz module. Let F be a field extension of $\mathbf{F}_q$ and $\chi\colon \Delta \to F^\times$ a homomorphism.

Let $\bar{F}$ be an algebraic closure of F and $\omega_1, \ldots, \omega_d$ the d distinct $\mathbf{F}_q$-embeddings of the field A/P in $\bar{F}$. Then χ can be uniquely written as

$$\chi = \omega_1^{s_1} \cdots \omega_d^{s_d}$$

with $0 \leq s_i \leq q-1$ for all i and not all s_i equal to $q-1$. Note that if we order the ω's so that $\omega_i = \omega_{i-1}^q$ for all i and if $\chi = \omega_1^n$ with $0 \leq n < q^d - 1$ then the s_i are the q-adic digits of n.

The *Gauss-Thakur sum* [15] associated with χ is defined as follows:

$$\tau(\chi) = \prod_{i=1}^{d} \left(- \sum_{\delta \in \Delta} \omega_i(\delta)^{-1} \otimes \delta(\lambda) \right)^{s_i} \in \bar{F} \otimes_{\mathbf{F}_q} \mathcal{O}_K. \tag{5}$$

One verifies that $\tau(\chi) \in F \otimes_{\mathbf{F}_q} \mathcal{O}_K$. Note that we have

$$\tau(\chi) = \prod_{i=1}^{d} \tau(\omega_i)^{s_i}.$$

We summarize the basic properties of these Gauss-Thakur sums:

Proposition 5.2. *Let F be an extension of $\mathbf{F}_q$ and $\chi\colon \Delta \to F^\times$ a homomorphism. Then*

(1) $\tau(\chi) \in e_\chi(F \otimes_{\mathbf{F}_q} \mathcal{O}_K)$;

(2) *if* $\chi \neq 1$ *then* $\tau(\chi)\tau(\chi^{-1}) = (-1)^d P$*;*
(3) $\tau(1) = 1$.

Proof. See [2, §2]. □

5.3. In particular, the proposition tells us that $\tau(\chi)$ is nonzero. Since $e_\chi(F \otimes_{\mathbf{F}_q} K)$ is free of rank one over $F \otimes_{\mathbf{F}_q} k$, we find that there is a unique $B_{1,\chi} \in F \otimes_{\mathbf{F}_q} k$ such that

$$e_\chi \frac{1}{\lambda} = B_{1,\chi}\tau(\chi)$$

in $F \otimes_{\mathbf{F}_q} K$. We will refer to these $B_{1,\chi}$ as *generalized Bernoulli-Carlitz numbers.*

5.4. For the trivial character $\chi = 1$ we have

$$B_{1,1} = e_\chi \frac{1}{\lambda} = -\operatorname{tr}_{K/k} \frac{1}{\lambda}.$$

Since the group $\mathbf{F}_q^\times$ acts freely on the set of conjugates of $1/\lambda$, we see that $B_{1,1} = 0$ if $q > 2$. If $q = 2$ then we have

$$B_{1,1} = \frac{P+1}{T^2+T}.$$

This follows from the easily proven fact that for any $Q \in \mathbf{F}_2[T]$ different from zero the Q-torsion of C is defined by a polynomial of the form

$$\varphi_Q(X) = QX + \frac{Q^2+Q}{T^2+T}X^2 + \cdots + X^{2^{\deg Q}}$$

in $k[X]$.

5.5. Now let $F/\mathbf{F}_q$ be an extension containing a field of q^d elements and consider

$$\eta = \sum_\chi \tau(\chi) \in F \otimes_{\mathbf{F}_q} \mathcal{O}_K$$

and

$$B_1 = \sum_\chi B_{1,\chi} e_\chi \in F \otimes_{\mathbf{F}_q} k[\Delta],$$

where the sums range over all homomorphisms $\Delta \to F^\times$. Then we have $\eta \in \mathcal{O}_K$ and $B_1 \in k[\Delta]$. They are related by the identity $\lambda^{-1} = B_1\eta$.

Theorem 5.6. $\mathcal{O}_K = A[\Delta]\eta$.

Proof. See [2, Théorème 2.5] or [5]. □

6. ∞-adic equivariant class number formula

In this section we prove Theorem A. The proof follows very closely the proof of the special value formula in [13], and rather than copying the whole proof, we give an overview of the argument, while treating in detail those parts that are different.

6.1. We start by giving an Euler product formula for the equivariant L-value $L(1,\Delta)$ defined in (4). If $\mathfrak{m}$ is a maximal ideal of A then both $\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K$ and $\mathrm{C}(\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K)$ are finite $A[\Delta]$-modules so we can consider the normalized generators $[\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K]_{A[\Delta]}$ respectively $[\mathrm{C}(\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K)]_{A[\Delta]}$ of their Fitting ideals, see 3.3.

Similarly, if F is an extension of $\mathbf{F}_q$ and M a finite $F \otimes_{\mathbf{F}_q} A$-module then we denote by $[M]_{F\otimes_{\mathbf{F}_q}A} \in F \otimes_{\mathbf{F}_q} A = F[T]$ the unique monic generator of the Fitting ideal of M.

Proposition 6.2. *Let F be an extension of $\mathbf{F}_q$ and $\chi\colon \Delta \to F^\times$ a homomorphism. Let $\mathfrak{m} \subset A$ be a maximal ideal, with monic generator f. Then we have*

$$\left[e_\chi(F \otimes_{\mathbf{F}_q} \mathrm{C}(\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K))\right]_{F\otimes_{\mathbf{F}_q} A} = f(T) - \chi(T).$$

Proof. Without loss of generality we may assume that F is algebraically closed. We need to show that

$$\det_{F[Z]}\left(Z - T - \tau \mid e_\chi\left(F \otimes_{\mathbf{F}_q} \frac{\mathcal{O}_K}{f\mathcal{O}_K}\right)[Z]\right) = f(Z) - \chi(f),$$

where τ is the $F[Z]$-linear map induced by the map $\mathcal{O}_K \to \mathcal{O}_K, x \mapsto x^q$. The module

$$M := e_\chi\left(F \otimes_{\mathbf{F}_q} \frac{\mathcal{O}_K}{f\mathcal{O}_K}\right)$$

is free of rank one over $F \otimes_{\mathbf{F}_q} A/fA \cong F^n$, with $n = \deg f$.

The F-linear action of τ on M permutes the n components cyclically. If f is coprime with P then we have that τ^n is the reduction of the Frobenius at f, hence τ^n acts as $\chi(f)$ on M. If $f = P$ then $\mathcal{O}_K/f\mathcal{O}_K \cong (A/P)[\epsilon]/\epsilon^{q^d-1}$ where τ^d acts as the identity on A/P and $\tau^d(\epsilon) = 0$, so we find that τ^n acts on M as

$$\chi(f) = \begin{cases} 0 & \text{if } \chi \neq 1 \\ 1 & \text{if } \chi = 1 \end{cases}$$

The action of T on M is diagonally by $(t_1, \cdots, t_n) \in F^n$ where the $t_i \in F$ are the roots of $f(T)$.

Combining these descriptions of the actions of T and τ we find that the characteristic polynomial of $T + \tau$ acting on M is $f(Z) - \chi(f)$, what we had to prove. □

By the Euler product formula

$$L(1, \chi) = \prod_f \left(1 - \frac{\chi(f)}{f}\right)^{-1}$$

with f running over the monic irreducible elements of A we conclude:

Corollary 6.3. *The infinite product*

$$\prod_{\mathfrak{m}} \frac{\left[\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K\right]_{A[\Delta]}}{\left[\mathrm{C}(\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K)\right]_{A[\Delta]}}$$

with $\mathfrak{m}$ ranging over the maximal ideals of A, converges in $k_\infty[\Delta]$ to $L(1, \Delta)$. □

6.4. Next we need a slight generalization of the trace formula of [13, §3]. Let F be a finite extension of $\mathbf{F}_q$. Let M be a free $F \otimes_{\mathbf{F}_q} A$-module of finite rank. Let $\tau\colon M \to M$ be an $\mathbf{F}_q$-linear map such that $\tau((x \otimes a)m) = (x \otimes a^q)\tau(m)$ for all $x \in F$, $a \in A$ and $m \in M$.

Let Ψ be a power series

$$\Psi = \sum_{i,j \geq 1} a_{ij} \tau^i Z^{-j}$$

with $a_{ij} \in A$ for all i, j, such that for all j there are only finitely many i with $a_{ij} \neq 0$. In other words, the coefficient of Z^{-j} is a polynomial in τ.

Then for every maximal ideal $\mathfrak{m}$ of A there is an obvious $F[[Z^{-1}]]$-linear action of Ψ on $F[[Z^{-1}]]\otimes_F(M/\mathfrak{m}M)$. Also, there is a natural $F[[Z^{-1}]]$-action of Ψ on the compact $F[[Z^{-1}]]$-module

$$F[[Z^{-1}]]\hat{\otimes}_F\frac{k_\infty\otimes_A M}{M}=\left\{\sum_{i\geq 0} m_i Z^{-i}\colon m_i\in\frac{k_\infty\otimes_A M}{M}\right\}$$

This endomorphism is *nuclear* in the sense of [13, §2], so we can take the determinant of $1+\Psi$ acting on this compact module.

Proposition 6.5. *The infinite product*

$$\prod_{\mathfrak{m}}\det\nolimits_{F[[Z^{-1}]]}\left(1+\Psi\,|\,F[[Z^{-1}]]\otimes_F\frac{M}{\mathfrak{m}M}\right)^{-1},$$

where $\mathfrak{m}$ *runs over the maximal ideals of* A*, converges to*

$$\det\nolimits_{F[[Z^{-1}]]}\left(1+\Psi\,|\,F[[Z^{-1}]]\hat{\otimes}_F\frac{k_\infty\otimes_A M}{M}\right).$$

Proof. The only difference with the formula of [13, §3] is that we deal with a q-Frobenius but with F-linear determinants for various finite extensions $F/\mathbf{F}_q$. However, the proof of this generalization is identical to the proof in [13]. □

Put

$$\Theta=\frac{1-(T+\tau)Z^{-1}}{1-TZ^{-1}}-1=-\sum_{n=1}^{\infty}\tau T^{n-1}Z^{-n}.$$

Applying the proposition with $\Psi=\Theta$ and $M=e_\chi(F\otimes_{\mathbf{F}_q}\mathcal{O}_K)$ for every $\chi\colon\Delta\to F^\times$ we get:

Proposition 6.6. *We have*

$$L(1,\Delta)=\det\nolimits_{\mathbf{F}_q[\Delta][[Z^{-1}]]}\left(1+\Theta\mid\mathbf{F}_q[[Z^{-1}]]\hat{\otimes}_{\mathbf{F}_q}\frac{K_\infty}{\mathcal{O}_K}\right)_{\big|Z=T}$$

in $k_\infty[\Delta]=\mathbf{F}_q[\Delta]((T^{-1}))$. □

We can now apply the same reasoning as in section §5 of [13]:

Proof of Theorem A. The exponential map induces a short exact sequence of compact $\mathbf{F}_q[\Delta]$-modules

$$0\to\frac{K_\infty}{\mathrm{U}}\overset{\exp}{\to}\frac{K_\infty}{\mathcal{O}_K}\to\mathrm{H}(\mathcal{O}_K)\to 0.$$

After the choice of a splitting, we obtain an isomorphism

$$\gamma\colon\frac{K_\infty}{\mathrm{U}}\times\mathrm{H}(\mathcal{O}_K)\to\frac{K_\infty}{\mathcal{O}_K}.$$

Since the map exp is infinitely tangent to the identity (in the sense of §4 of [13]), and since

$$1+\Theta=\frac{1-\gamma T\gamma^{-1}Z^{-1}}{1-TZ^{-1}}$$

we conclude using [13, Theorem 4] that

$$\left(1+\Theta\mid\mathbf{F}_q[[Z^{-1}]]\hat{\otimes}_{\mathbf{F}_q}\frac{K_\infty}{\mathcal{O}_K}\right)_{\big|Z=T}=\left[\mathrm{H}(\mathcal{O}_K)\right]_{A[\Delta]}\left[\mathcal{O}_K:\mathrm{U}(\mathcal{O}_K)\right]_{A[\Delta]}$$

which proves the theorem. ☐

7. Cyclotomic units

This section is based on Anderson's fundamental paper [1], in which he explicitly constructed a finitely generated submodule of $\mathrm{C}(\mathcal{O}_K)$ and related it to the special values $L(1,\chi)$ and $L_P(1,\chi)$. We bypass some of Anderson's proofs by using the equivariant class number formula of the preceding section.

7.1. Let $\lambda \in K$ be a generator of the P-torsion of the Carlitz module. For all $m \geq 0$ define

$$\mathfrak{L}_m := \sum_{\sigma\in\Delta} \sigma(\lambda)^m \sum_{a\in A_{+,\sigma}} \frac{1}{a} \in K_\infty \tag{6}$$

where $A_{+,\sigma}$ is the set of monic elements of A that are congruent to σ in A/P. Let $\mathfrak{M} \subset K_\infty$ be the A-module generated by all the $\mathfrak{L}_m$.

Proposition 7.2. *For all $\sigma \in \Delta$ we have $\sigma\mathfrak{M} = \mathfrak{M}$.*

Proof. Let $\sigma \in \Delta$ and $m \geq 0$. We need to show that $\sigma\mathfrak{L}_m \in \mathfrak{M}$. We have $\sigma(\lambda^m) \in \mathcal{O}_K = A[\lambda]$, hence there are $a_i \in A$ so that $\sigma(\lambda^m) = \sum a_i\lambda^i$. But then we have $\sigma(\mathfrak{L}_m) = \sum a_i \mathfrak{L}_i \in \mathfrak{M}$, as desired. ☐

Proposition 7.3. *Let F be a field extension of $\mathbf{F}_q$ and $\chi\colon \Delta \to F^\times$ a homomorphism. Then we have*

$$e_\chi(F \otimes_{\mathbf{F}_q} \mathfrak{M}) = L(1,\chi)\cdot e_\chi(F\otimes_{\mathbf{F}_q} \mathcal{O}_K)$$

as sub-$F \otimes_{\mathbf{F}_q} A$-modules of $e_\chi(F\otimes_{\mathbf{F}_q} K_\infty)$.

Proof. For $\sigma \in \Delta$ we have

$$e_\chi \sigma(\lambda)^m = \chi(\sigma) e_\chi \lambda^m$$

hence

$$e_\chi \mathfrak{L}_m = \left(\sum_{\sigma\in\Delta} \sum_{a\in A_{+,\sigma}} \frac{\chi(\sigma)}{a} \right) e_\chi\lambda^m = L(1,\chi) e_\chi \lambda^m.$$

In particular, we have that $e_\chi(F\otimes_{\mathbf{F}_q}\mathfrak{M})$ is generated by

$$\{L(1,\chi)e_\chi\lambda^m \colon m \geq 0\} \subset e_\chi(F\otimes_{\mathbf{F}_q} K_\infty) \tag{7}$$

as an $F\otimes_{\mathbf{F}_q} A$-module. Because $\mathcal{O}_K = A[\lambda]$ (see 4.2) we also have that $e_\chi(F\otimes_{\mathbf{F}_q}\mathcal{O}_K)$ is generated by

$$\{e_\chi\lambda^m \colon m\geq 0\} \subset e_\chi(F\otimes_{\mathbf{F}_q} K_\infty) \tag{8}$$

as an $F\otimes_{\mathbf{F}_q} A$-module. Comparing the generating sets (7) and (8) we obtain the proposition. ☐

Assembling the isotypical components together we obtain

Theorem 7.4. $\mathfrak{M} = L(1,\Delta)\cdot\mathcal{O}_K$ *as $A[\Delta]$-submodules of K_∞.* ☐

In particular we have:

Corollary 7.5. $\mathfrak{M}$ *is free of rank one over $A[\Delta]$.* ☐

Comparing Theorems A and 7.4 leads to:

Corollary 7.6. *$\mathfrak{M} \subset \mathrm{U}(\mathcal{O}_K)$ and the $A[\Delta]$-modules $\mathrm{H}(\mathcal{O}_K)$ and $\mathrm{U}(\mathcal{O}_K)/\mathfrak{M}$ have the same Fitting ideal.* □

Finally we see that exponentiating the generators of $\mathfrak{M}$ indeed yields integral points on the Carlitz module:

Corollary 7.7. *$\exp \mathfrak{M} \subset \mathcal{U}$.* □

8. Odd part of $\mathrm{H}(\mathcal{O}_K)$

8.1. Fix a place v of K above ∞ and a generator $\bar{\pi} \in K_v$ of the kernel of $\exp_{\mathrm{C}}\colon K_v \to \mathrm{C}(K_v)$. Put $\lambda = \exp_{\mathrm{C}}(\bar{\pi}/P)$. Then λ lies in $K \subset K_v$ and is a generator of $C[P](K)$.

Proposition 8.2. *Let F be a field containing $\mathbf{F}_q$ and let $\chi\colon \Delta \to F^\times$ be odd. If $\chi \neq 1$ then*

$$L(1,\chi) = \frac{\bar{\pi} B_{1,\chi^{-1}} \tau(\chi^{-1})}{P}$$

in $F \otimes_{\mathbf{F}_q} K_v$. If $\chi = 1$ then $q = 2$ and

$$L(1,\chi) = \frac{\bar{\pi}}{T^2 + T}$$

in $F \otimes_{\mathbf{F}_q} k_\infty$.

If χ extends to a ring homomorphism $A/P \to F$ then a similar formula for $L(1,\chi)$ has been obtained by Pellarin [9, Corollary 2].

Proof of Proposition 8.2. Take the logarithmic derivative of both sides in the product expansion

$$\exp_C X = X \prod_{a \in A \setminus \{0\}} \left(1 - \frac{X}{a\bar{\pi}}\right)$$

in $K_v[[X]]$ to find

$$\frac{1}{\exp_C X} = \frac{1}{X} + \sum_{a \in A \setminus \{0\}} \frac{1}{X - a\bar{\pi}} = \sum_{a \in A} \frac{1}{X + a\bar{\pi}}.$$

Let $b \in A$ be coprime with P and denote by σ_b its image in Δ. Substituting $X = \frac{b}{P}\bar{\pi}$ we obtain

$$\frac{1}{\sigma_b(\lambda)} = \sum_{a \in A} \frac{1}{(a + \frac{b}{P})\bar{\pi}} = \frac{P}{\bar{\pi}} \sum_{a \equiv b \ (P)} \frac{1}{a}. \tag{9}$$

Now assume $\chi \neq 1$. Multiplying both sides in (9) with $\chi(b)$ and summing over all classes of b in $\Delta = (A/P)^\times$ we find

$$e_{\chi^{-1}} \frac{1}{\lambda} = -\frac{P}{\bar{\pi}} \sum_{a \in A} \frac{\chi(a)}{a} = \frac{P}{\bar{\pi}} \sum_{a \in A_+} \frac{\chi(a)}{a} = L(1,\chi) \frac{P}{\bar{\pi}}$$

in $F \otimes_{\mathbf{F}_q} K_v$, where in the middle equality we have used that χ is odd. By 5.3 we conclude

$$B_{1,\chi^{-1}} \tau(\chi^{-1}) = L(1,\chi) \frac{P}{\bar{\pi}}$$

in $F \otimes_{\mathbf{F}_q} K_v$, what we had to prove.

If $\chi = 1$ then summing (9) over all b gives

$$\mathrm{tr}_{K/k}\frac{1}{\lambda} = \frac{P}{\bar{\pi}}L(1,\chi)\frac{P-1}{P}$$

(the last factor compensates for the extra Euler factor at P in $L(1,\chi)$). Using 5.4 we conclude $L(1,\chi) = \bar{\pi}/(T^2+T)$, as claimed. □

8.3. Let v and $\bar{\pi} \in K_v$ be as in 8.1. Let

$$\bar{\pi}_v = (0,\dots,0,\bar{\pi},0,\dots 0) \in K_\infty$$

be the image of $\bar{\pi}$ under the inclusion $K_v \to K_\infty$.

Proposition 8.4. *Λ is a free rank one $A[\Delta]^-$-module, generated by $\bar{\pi}_v$.*

Proof. Clearly Λ is generated by $\{\sigma(\bar{\pi}_v)\colon \sigma \in \Delta\}$ as an A-module, and since $\Lambda^+ = 0$ (see 4.6) we find that $\Lambda = A[\Delta]^-\bar{\pi}_v$. Both Λ and $A[\Delta]^-$ are free of rank $(q^d-1)/(q-1)$ over A so we conclude that Λ is the free $A[\Delta]^-$- module generated by $\bar{\pi}_v$. □

Proposition 8.5. *If $\chi\colon \Delta \to F^\times$ is odd and $\chi \neq 1$ then*

$$L(1,\chi)e_\chi(F \otimes_{\mathbf{F}_q} \mathcal{O}_K) = B_{1,\chi^{-1}}e_\chi(F \otimes_{\mathbf{F}_q} \Lambda)$$

in $F \otimes_{\mathbf{F}_q} K_\infty$.

Proof. Both sides are free $F \otimes_{\mathbf{F}_q} A$-modules of rank one. The left-hand-side is generated by

$$L(1,\chi)\tau(\chi) \in F \otimes_{\mathbf{F}_q} K_\infty$$

and by Proposition 8.4 the right-hand-side is generated by

$$B_{1,\chi^{-1}}e_\chi\bar{\pi}_v \in F \otimes_{\mathbf{F}_q} K_\infty.$$

Let α be the quotient of these generators:

$$\alpha := \frac{B_{1,\chi^{-1}}e_\chi\bar{\pi}_v}{L(1,\chi)\tau(\chi)} \in (F \otimes_{\mathbf{F}_q} K_\infty)^\times.$$

We need to show $\alpha \in F^\times$. Since α is Δ-invariant, we have $\alpha \in F \otimes_{\mathbf{F}_q} k_\infty$ and it suffices to show that the v-component α_v of α is in $F^\times$. Using that χ is odd we find

$$\alpha_v = \frac{B_{1,\chi^{-1}}\bar{\pi}}{L(1,\chi)\tau(\chi)} \in F \otimes_{\mathbf{F}_q} K_v,$$

and by Proposition 8.2

$$\alpha_v = \frac{P}{\tau(\chi^{-1})\tau(\chi)}.$$

Using Proposition 5.2 we conclude $\alpha_v = (-1)^d$ and therefore $\alpha \in F^\times$. □

Lemma 8.6. $\mathcal{U}^- = \mathcal{U}_{\mathrm{tors}} = \mathrm{C}(\mathcal{O}_K)_{\mathrm{tors}}$.

Proof. By 4.5 we obtain a short exact sequence of $A[\Delta]^-$-modules

$$0 \to \Lambda \to \mathrm{U}(\mathcal{O}_K)^- \to \mathcal{U}^- \to 0$$

and since $\mathrm{U}(\mathcal{O}_K)$ is free of rank one over $A[\Delta]$, we find that Λ and $\mathrm{U}(\mathcal{O}_K)^-$ have the same A-rank. We conclude that $\mathcal{U}^-$ is torsion. Since $\Lambda^+ = 0$, the module $\mathcal{U}^+$ is torsion-free, so $\mathcal{U}^- = \mathcal{U}_{\mathrm{tors}}$. In [12, Prop. 2] it is shown that $\mathrm{C}(\mathcal{O}_K)_{\mathrm{tors}} \subset \mathcal{U}$, so we conclude $\mathcal{U}_{\mathrm{tors}} = \mathrm{C}(\mathcal{O}_K)_{\mathrm{tors}}$. □

We can now prove Theorem B:

Theorem 8.7. *Let F be a field containing $\mathbf{F}_q$ and let $\chi\colon \Delta \to F$ be an odd character. Consider the ideal $I = \operatorname{Fitt} e_\chi(F \otimes_{\mathbf{F}_q} \mathrm{H}(\mathcal{O}_K))$ in $F \otimes_{\mathbf{F}_q} A$. Then*

(1) $I = (1)$ *if* $\chi = 1$ *(and then* $q = 2$*);*
(2) $I = ((T - \chi(T))B_{1,\chi^{-1}})$ *if* χ *extends to a ring homomorphism* $A/P \to F$*;*
(3) $I = (B_{1,\chi^{-1}})$ *otherwise.*

Proof. Let S denote the set of $\chi\colon \Delta \to F^\times$ that extend to a ring homomorphism $A/P \to F$.

The equivariant class number formula (Theorem A) says that

$$L(1,\chi)\tau(\chi)\; F \otimes_{\mathbf{F}_q} A = I \cdot e_\chi(F \otimes_{\mathbf{F}_q} \mathrm{U}(\mathcal{O}_K)) \tag{10}$$

in $F \otimes_{\mathbf{F}_q} K_\infty$. The preceding lemma gives us a short exact sequence

$$0 \to \Lambda \to \mathrm{U}(\mathcal{O}_K)^- \to \mathrm{C}(K)_{\mathrm{tors}} \to 0,$$

from which we get

$$e_\chi(F \otimes_{\mathbf{F}_q} \mathrm{U}(\mathcal{O}_K))) = \begin{cases} (T^2+T)^{-1} e_\chi(F \otimes_{\mathbf{F}_q} \Lambda) & \text{if } \chi = 1, \\ (T-\chi(T))^{-1} e_\chi(F \otimes_{\mathbf{F}_q} \Lambda) & \text{if } \chi \in S, \\ e_\chi(F \otimes_{\mathbf{F}_q} \Lambda) & \text{otherwise.} \end{cases}$$

If $\chi \neq 1$ then the theorem follows from (10) and Proposition 8.5. If $\chi = 1$ then (10) gives

$$L(1,\chi)A = I(T^2+T)^{-1}\bar{\pi}$$

in k_∞, and the theorem follows from Proposition 8.2. □

8.8. Next we will prove congruences modulo P between the generalized Bernoulli-Carlitz numbers $B_{1,\chi}$ and the usual Bernoulli-Carlitz numbers BC_n. We then use these congruences to give a new proof of the Herbrand-Ribet theorem of [14], based on Theorem 8.7.

8.9. Let n be a non-negative integer with q-adic expansion

$$n = n_0 + n_1 q + n_2 q^2 + \cdots, \quad 0 \le n_i < q.$$

For all $i \ge 0$ let

$$D_i = \prod_{j=0}^{i-1} (T^{q^i} - T^{q^j}).$$

The n-th *Carlitz factorial* $\Pi(n)$ is defined to be

$$\Pi(n) := \prod_{i \ge 0} D_i^{n_i} \in A.$$

Note that $v_P(\Pi(n)) = 0$ for all $n < q^d$.

8.10. For all $n \ge 0$ the *Bernoulli-Carlitz numbers* $\mathrm{BC}_n \in k$ are defined by the power series identity

$$\frac{X}{\exp X} = \sum_{n \ge 0} \mathrm{BC}_n \frac{X^n}{\Pi(n)} \in k[[X]].$$

8.11. *Convention.* The completions k_P and $\mathcal{O}_{K,P}$ are naturally A/P-algebras. We have a canonical idenfitication

$$\mathrm{Hom}(\Delta, (A/P)^\times) = \mathrm{Hom}(\Delta, A_P^\times).$$

For a $\chi\colon \Delta \to (A/P)^\times$ we will denote by $B_{1,\chi}$ and $\tau(\chi)$ the images of $B_{1,\chi}$ and $\tau(\chi)$ under the natural maps

$$A/P \otimes_{\mathbf{F}_q} k \to k_P$$

and

$$A/P \otimes_{\mathbf{F}_q} \mathcal{O}_K \to \mathcal{O}_{K,P}$$

respectively.

Proposition 8.12. *Let n be an integer with $0 \leq n < q^d - 1$. Then in $\mathcal{O}_{K,P}$ we have the congruence*

$$\tau(\omega^n) \equiv \frac{\lambda^n}{\Pi(n)} \pmod{\mathfrak{m}^{n+1}}.$$

Proof. Writing n in its q-adic expansion, we see from the definitions of $\tau(\omega^n)$ and $\Pi(n)$ that it suffices to prove

$$\tau(\omega^{q^i}) \equiv \frac{\lambda^{q^i}}{D_i} \pmod{\mathfrak{m}^{q^i+1}}$$

for all i satisfying $0 \leq i < d$. This is shown in [15, Theorem VI]. Note that Thakur's notation is different from ours. His λ is congruent to our λ modulo $\mathfrak{m}^2$, but not necessarily the same (see [15, Lemma II]). Also note that there is a typo in the proof of Theorem VI of *loc. cit.*: the left-hand side of the displayed formula should be g_j/λ^{q^j} rather than g_j/λ^{q^h}. □

Theorem 8.13. *If $n \neq 1$ and $0 < n \leq q^d - 1$ then $B_{1,\omega^{-n}} \in A_P$ and*

$$B_{1,\omega^{-n}} \equiv \frac{\Pi(q^d-1-n)}{\Pi(q^d-n)} \mathrm{BC}_{q^d-n}$$

modulo P.

Proof. (Compare with §8 of [14].) Consider the exponential power series

$$\exp_{\mathrm{C}} X = X + e_1 X^q + \cdots \in k[[X]].$$

The coefficients $e_1, \cdots, e_{d-1}$ are P-integral, so we can construct the truncated reduced exponential

$$\overline{\exp}_{\mathrm{C}} X = X + e_1 X^q + \cdots + e_{d-1} X^{q^{d-1}} \in (A/P)[[X]]/(X^{q^d}).$$

This defines a map $\overline{\exp}_{\mathrm{C}}\colon \mathfrak{m}/\mathfrak{m}^{q^d} \to \mathfrak{m}/\mathfrak{m}^{q^d}$ which is an isomorphism since it induces the identity map on the intermediate quotients $\mathfrak{m}^i/\mathfrak{m}^{i+1}$.

Put $\bar\beta := \overline{\exp}_{\mathrm{C}}^{-1}\lambda$ and let $\beta \in \mathfrak{m}$ be a lift of $\bar\beta$. Then we have

$$\frac{1}{\lambda} \equiv \sum_{n=0}^{q^d-1} \frac{\mathrm{BC}_n}{\Pi(n)} \beta^{n-1} \pmod{\mathfrak{m}^{q^d-1}}. \tag{11}$$

Moreover, by [14, Lemma 4] we have for $n, m \in \{0, \ldots, q^d-2\}$

$$e_{\omega^n}\beta^m \equiv 0 \pmod{\mathfrak{m}^{q^d}} \quad \text{if } n \neq m$$

and
$$e_{\omega^n}\beta^m \equiv \beta^m \pmod{\mathfrak{m}^{q^d}} \quad \text{if } n = m.$$

Applying $e_{\omega^{-n}} = e_{\omega^{q^d-1-n}}$ to (11) we obtain
$$e_{\omega^{-n}}\lambda^{-1} \equiv \frac{\mathrm{BC}_{q^d-n}}{\Pi(q^d-1-n)}\beta^{q^d-1-n} \pmod{\mathfrak{m}^{q^d-1}}$$
and therefore
$$\tau(\omega^n)B_{1,\omega^{-n}} \equiv \mathrm{BC}'_{q^d-n}\,\beta^{q^d-1-n} \pmod{\mathfrak{m}^{q^d-1}}.$$
Together with Proposition 8.12 this proves the Theorem. □

If we combine Theorem 8.7 with the congruence of Theorem 8.13 we obtain a new proof of the Herbrand-Ribet theorem of [14]:

Theorem 8.14. *Let $1 < n < q^d - 1$ be divisible by $q - 1$. Then*
$$e_{\omega^{1-n}}(A/P \otimes_{\mathbf{F}_q} \mathrm{H}(\mathcal{O}_K))$$
is non-zero if and only if $v_P(\mathrm{BC}_n) > 0$.

Proof. Passing from A to A_P in Theorem 8.7 and splitting out character by character we find

(12) $$\mathrm{length}_{A_P}\, e_\chi(A_P \otimes_A \mathrm{H}(\mathcal{O}_K)) = v_P(B_{1,\chi^{-1}}) + \mathrm{length}_{A_P}\, e_\chi \mathrm{C}(K)_{\mathrm{tors}}$$

for all odd $\chi\colon \Delta \to A_P^\times$.

Recall from 4.7 that $\mathrm{C}(\mathcal{O}_K)_{\mathrm{tors}} \cong A/Q$ with
$$Q = \begin{cases} P & \text{if } q = 2 \\ \mathrm{lcm}(P, T(T+1)) & \text{if } q > 2 \end{cases},$$
where the action of $\Delta = (A/P)^\times = (A/Q)^\times$ on A/Q is the tautological one. In particular, $A_P \otimes_A \mathrm{C}(K)_{\mathrm{tors}} = A/P$ and

(13) $$e_\chi A_P \otimes_A \mathrm{C}(K)_{\mathrm{tors}} \cong \begin{cases} A/P & \text{if } \chi = \omega \\ 0 & \text{if } \chi \neq \omega \end{cases}$$

for all $\chi\colon \Delta \to A_P^\times$.

Combining (12) and (13) we find
$$\mathrm{length}_{A_P}\, e_\chi(A_P \otimes_A \mathrm{H}(\mathcal{O}_K)) = v_P(B_{1,\chi^{-1}})$$
for all even $\chi\colon \Delta \to A_P^\times$, different from ω. Now by Theorem 8.13 we have
$$v_P(B_{\omega^{1-n}}) > 0 \Longleftrightarrow v_P(\mathrm{BC}'_n) > 0,$$
which concludes the proof. □

9. Even part of $\mathrm{H}(\mathcal{O}_K)$

9.1. Let $\mathcal{L} \subset \mathrm{C}(\mathcal{O}_K)$ be the image of $\mathfrak{M}$ in $\mathrm{C}(\mathcal{O}_K)$ and $\sqrt{\mathcal{L}}$ its divisible closure in $\mathrm{C}(\mathcal{O}_K)$, that is,
$$\sqrt{\mathcal{L}} = \{m \in \mathrm{C}(\mathcal{O}_K) \colon \exists a \in A \setminus \{0\} \text{ such that } am \in \mathcal{L}\}.$$

Proposition 9.2. $\sqrt{\mathcal{L}} = \mathcal{U}$.

Proof. See the remark after [12, Prop. 2]. □

Theorem 9.3. $\mathrm{Fitt}_{A[\Delta]}\, \mathcal{U}/\mathcal{L} = \mathrm{Fitt}_{A[\Delta]}\, \mathrm{H}(\mathcal{O}_K)^+$.

Proof. The minus-part of the short exact sequence of $A[\Delta]$-modules

$$0 \to \mathfrak{M} \cap \ker \exp_C \to \mathfrak{M} \to \mathcal{L} \to 0$$

is the short exact sequence

$$0 \to \mathfrak{M} \cap \ker \exp_C \to \mathfrak{M}^- \to \mathrm{C}[P](K) \to 0.$$

Similarly, the minus-part of the short exact sequence

$$0 \to \ker \exp_C \to \mathrm{U}(\mathcal{O}_K) \to \mathcal{U} \to 0$$

is the short exact sequence

$$0 \to \ker \exp_C \to \mathrm{U}(\mathcal{O}_K)^- \to \mathrm{C}[P](K) \to 0.$$

Comparing both, we find

$$\mathrm{Fitt}_{A[\Delta]} \frac{\mathcal{U}}{\mathcal{L}} = \mathrm{Fitt}_{A[\Delta]} \frac{\mathcal{U}^+}{\mathcal{L}^+} = \mathrm{Fitt}_{A[\Delta]} \frac{\mathrm{U}(\mathcal{O}_K)^+}{\mathfrak{M}^+}.$$

The ideal on the left equals $\mathrm{Fitt}_{A[\Delta]} \sqrt{\mathcal{L}}/\mathcal{L}$ by Proposition 9.2, and the ideal on the right equals $\mathrm{Fitt}_{A[\Delta]} \mathrm{H}(\mathcal{O}_K)^+$ by Corollary 7.6. □

In [3] we have shown that $A_P \otimes_A \mathrm{H}(\mathcal{O}_K)^+$ is not always trivial, unlike what one may expect by analogy with the Kummer-Vandiver conjecture. Combining this with Theorem 9.3 we conclude

Corollary 9.4. *There exist prime powers q and monic irreducible $P \in \mathbf{F}_q[T]$ so that $\sqrt{\mathcal{L}}/\mathcal{L}$ has nontrivial P-torsion.* □

This settles Anderson's conjecture [1, §4.12] in the negative. For example, the prime

$$P = T^9 - T^6 - T^4 - T^3 - T^2 + 1 \;\in\; \mathbf{F}_3[T]$$

gives a counterexample [3].

9.5. We now turn our attention to P-adic units. Let $\mathcal{U}$ be the image of $\mathrm{U}(\mathcal{O}_K)$ in $\mathrm{C}(\mathcal{O}_{K,P})$ and put

$$\mathcal{U}' := \mathcal{U} \cap \mathfrak{m}.$$

Then $\mathcal{U}'$ is a sub-A-module of finite index in $\mathcal{U}$. We denote by $\bar{\mathcal{U}}$ the closure of $\mathcal{U}'$ in $\mathrm{C}(\mathcal{O}_{K,P})$. This is an A_P-module. The natural map

$$\alpha \colon A_P \otimes_A \mathcal{U}' \to \bar{\mathcal{U}}$$

is surjective. We will now show that α is an isomorphism, a statement analogous to Leopoldt's conjecture for cyclotomic number fields (a theorem by Brumer [4]). The main point of the argument is a result on linear independence of P-adic Carlitz logarithms, which is shown by Vincent Bosser in an appendix to this paper.

Theorem 9.6. *α is an isomorphism.*

Corollary 9.7. *$\bar{\mathcal{U}}$ is free of rank one over $A_P[\Delta]^+$.* □

Proof of Theorem 9.6. For all $\chi\colon \Delta \to (A/P)^\times$ which are even, we show that $\bar{\mathcal{U}}(\chi)$ is free of rank one over A_P.

The Carlitz exponential defines an isomorphism of A_P-modules

$$\exp_{\mathrm{C}} \colon \mathfrak{m} \to \mathrm{C}(\mathfrak{m}).$$

Let $\log\mathcal{U}'$ be the inverse image of $\mathcal{U}' \subset \mathrm{C}(\mathfrak{m})$. By the Baker-Brumer theorem of Vincent Bosser (see appendix) the natural map

$$A/P \otimes_{\mathbf{F}_q} \log\mathcal{U}' \to (A/P) \cdot \log\mathcal{U}'$$

is an isomorphism. (In the target of this map we consider A/P as a subring of A_P). Now let $\chi\colon \Delta \to (A/P)^\times$ be an even character. Then

$$e_\chi(A/P \otimes_{\mathbf{F}_q} \mathcal{U}')$$

is nonzero, and hence its image in $\log\mathcal{U}'$ is nonzero. Since $e_\chi\mathfrak{m}$ is free of rank one over A_P, the theorem follows. □

Let $\bar{\mathcal{L}} \subset \mathrm{C}(\mathcal{O}_{K,P})$ be the topological closure of $\mathcal{L} \cap \mathfrak{m}$. Then $\bar{\mathcal{L}}$ is an $A_P[\Delta]$-module and $\bar{\mathcal{L}} \subset \bar{\mathcal{U}}$ with finite quotient.

Proposition 9.8. *Let* $\chi\colon \Delta \to A_P^\times$ *be a homomorphism. Then*

$$e_\chi\bar{\mathcal{L}} = L_P(1,\chi) \cdot e_\chi \mathrm{C}(\mathfrak{m})$$

as A_P*-submodules of* $\mathrm{C}(\mathfrak{m})$.

Proof. For $m \geq 1$ consider the series

$$\mathfrak{L}_{m,P} := \sum_{\sigma\in\Delta} \sigma(\lambda)^m \left(\sum_{n\geq 0} \sum_{a \in A_{+,n,\sigma}} \frac{1}{a} \right).$$

Here $A_{+,n,b}$ is the set of monic polynomials in A of degree n which reduce modulo P to $\sigma \in (A/P)^\times$. By [1, Proposition 12] this series converge P-adically to an element of $\mathfrak{m}$ and we have the remarkable identity

$$\exp_{\mathrm{C}} \mathfrak{L}_{m,P} = \exp_{\mathrm{C}} \mathfrak{L}_m \quad \text{for all } m \geq 1.$$

Note that this is an identity in $\mathrm{C}(\mathcal{O}_K)$, but that a priori the left-hand side is P-adic and lives in $\mathrm{C}(\mathfrak{m})$ whereas the right hand-side is ∞-adic and lives in $\mathrm{C}(K_\infty)$.

We have

$$\sum_{m\geq 1} A \exp_{\mathrm{C}} \mathfrak{L}_m \subset \mathcal{L} \cap \mathfrak{m}$$

and by [1, Proposition 9] the quotient is annihilated by $P-1$. Taking topological closures we find

$$\bar{\mathcal{L}} = \sum_{m\geq 1} A_P \exp_C \mathfrak{L}_{m,P}$$

as A_P-modules.

By exactly the same reasoning as in Proposition 7.3 we have for all $m \geq 1$ and for all $\chi\colon \Delta \to A_P^\times$ that

$$e_\chi \mathfrak{L}_{m,P} = L_P(1,\chi) e_\chi \lambda^m.$$

Since the A_P-module $C(\mathfrak{m})$ is generated by $(\lambda^m)_{m\geq 1}$ and since $\exp_{\mathrm{C}}$ defines an isomorphism $\mathfrak{m} \to C(\mathfrak{m})$ of $A_P[\Delta]$-modules, we conclude

$$e_\chi \bar{\mathcal{L}} = L_P(1,\chi) e_\chi \mathrm{C}(\mathfrak{m}).$$ □

Corollary 9.9. $L_P(1,\chi) = 0$ *if and only if* χ *is odd.* □

Now for even χ by Theorem 9.3 we have

$$\mathrm{length}_{A_P}\, e_\chi(A_P \otimes_A \mathrm{H}(\mathcal{O}_K)) = \mathrm{length}_{A_P} \frac{e_\chi \bar{\mathcal{U}}}{e_\chi \bar{\mathcal{L}}}.$$

Together with the above proposition this proves Theorem D:

Theorem 9.10. *Let $\chi\colon \Delta \to A_P^\times$ be even. Then $L_P(1,\chi) \neq 0$ and*

$$\mathrm{length}_{A_P}\, e_\chi\, (A_P \otimes_A \mathrm{H}(\mathcal{O}_K)) + \mathrm{length}_{A_P}\, e_\chi \frac{\mathrm{C}(\mathfrak{m})}{\bar{\mathcal{U}}} = v_P(L_P(1,\chi)).$$

□

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References

[1] G. Anderson, Log-Algebraicity of Twisted A-Harmonic Series and Special Values of L-Series in Characteristic p, *J. Number Theory* **60** (1996), 165–209.
[2] B. Anglès, Bases normales relatives en caractéristique positive, *J. Théor. Nombres Bordeaux* **14** (2002), no. 1, 1–17.
[3] B. Anglès and L. Taelman, On a problem à la Kummer-Vandiver for function fields, to appear in *J. Number Theory* (2012).
[4] A. Brumer, On the units of algebraic number fields, *Mathematika* **14** (1967), 121–124.
[5] R. J. Chapman, Carlitz modules and normal integral bases, *J. London Math. Soc.* **44** (1991), 250–260.
[6] D. Goss, L-series of t-motives and Drinfeld modules, in *The Arithmetic of Function Fields* (G. Goss, D. R. Hayes, and M. I. Rosen, Eds.), pp. 313–402, de Gruyter, 1992.
[7] D. Goss, *Basic Structures of Function Field Arithmetic*, Springer, 1996.
[8] B. Mazur and A. Wiles, Class fields of abelian extensions of $\mathbf{Q}$, *Invent. Math.* **76** (1984), 179–330.
[9] F. Pellarin, Values of certain L-series in positive characteristic, preprint (2011), `http://arxiv.org/abs/1107.4511`. To appear in *Ann. of Math.*
[10] B. Poonen, Local height functions and the Mordell-Weil theorem for Drinfel'd modules, *Compositio Math.* **97** (1995), no. 3, 349–368.
[11] M. Rosen, *Number theory in function fields*, Springer, 2002.
[12] L. Taelman, A Dirichlet unit theorem for Drinfeld modules, *Math. Ann.* **348** (2010), 899–907.
[13] L. Taelman, Special L-values of Drinfeld modules, *Ann. of Math.* **175** (2012), 369–391.
[14] L. Taelman, A Herbrand-Ribet theorem for function fields, *Invent. Math.* **188** (2012), 253–275.
[15] D. Thakur, Gauss sums for $\mathbf{F}_q[T]$, *Invent. Math.* **94** (1988), 105–112.

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APPENDIX:
A P-ADIC BAKER'S THEOREM FOR CARLITZ LOGARITHMS

VINCENT BOSSER

1. Notation and statement of the Theorem

We denote by $\mathbf{N} = \{0, 1, \ldots\}$ the set of nonnegative integers. We write $A = \mathbf{F}_q[T]$ and $k = \mathbf{F}_q(T)$. Let $P \in A$ be an irreducible polynomial of degree $d \geq 1$, let k_P be the completion of k at P, let $\mathbf{C}_P$ be the completion of an algebraic closure of k_P, and let $\bar{k} \subset \mathbf{C}_P$ be the algebraic closure of k in $\mathbf{C}_P$. We denote by $v = v_P$ the valuation on $\mathbf{C}_P$ corresponding to P normalized by $v_P(P) = 1$, and by $|\cdot| = |\cdot|_P$ the absolute value on $\mathbf{C}_P$ defined by $|z| = q^{-v_P(z)}$. We denote by $\Phi : A \to k\{\tau\}$ the Carlitz module and by

$$e(X) = \sum_{i \geq 0} \frac{X^{q^i}}{D_i} \in k[[X]] \tag{1}$$

the Carlitz exponential series. Let $\rho := q^{-1/(q^d-1)}$ be the convergence radius of the series (1), and put

$$D_\rho := \{z \in \mathbf{C}_P \mid |z| < \rho\}.$$

We know that $e(z)$ is convergent in $\mathbf{C}_P$ if and only if $z \in D_\rho$, and that the series (1) induces a bijection (the P-adic Carlitz exponential)

$$e : D_\rho \to D_\rho.$$

The inverse map will be denoted by Log (P-adic Carlitz logarithm).

The functions e and Log satisfy the following properties:

$$\forall z \in D_\rho,\ |e(z)| = |z|,$$

$$\forall a \in A, \forall z \in D_\rho,\ e(az) = \Phi_a(e(z)),$$

and

$$\forall a \in A, \forall z \in D_\rho,\ \operatorname{Log}(\Phi_a(z)) = a \operatorname{Log}(z). \tag{2}$$

The aim of this appendix is to give a proof of the following theorem:

Theorem 1. *Let $n \geq 1$, and let $\alpha_1, \ldots, \alpha_n$ be n elements of $D_\rho \cap \bar{k}$. If the numbers $\operatorname{Log} \alpha_1, \ldots, \operatorname{Log} \alpha_n$ are linearly independent over k, then they are linearly independent over $\bar{k}$.*

This theorem is an analogue of a theorem of Brumer for p-adic logarithms in characteristic zero [3], which was itself an analogue of a theorem of Baker [1] for usual complex logarithms. Many generalizations and improvements of the results of Baker are known nowadays in characteristic zero. The interested reader might consult *e.g.* the first three chapters of the book [9] as well as [8], [7], [4] for an overview of known results.

In the framework of Drinfeld modules, it seems that only results for logarithms in $\mathbf{C}_\infty$ (where $\infty = 1/T$) have been published yet. The first analogue of Baker's theorem is due to Yu [10] for an arbitrary Drinfeld module defined over $\bar{k}$, but under a separability condition. This condition was removed for CM-Drinfeld modules in [6], and then in full generality independently in [5] and [11]. A (quantitative) generalization of these results is established in [2].

Here we will consider for simplicity Carlitz-logarithms, and we will prove only a qualitative version of the so-called "homogeneous case" of the P-adic Baker's theorem. There is no doubt that one could obtain quantitative and nonhomogeneous[1] results for an arbitrary Drinfeld module defined over $\bar{k}$, *e.g.* using the methods of [11] or [2]. However, the proof would be much more complicated.

We will give here a proof of Theorem 1 as self-contained as possible. We will follow the exposition given in characteristic zero in [8, Section 6.3]. This proof is very close to the proof of [6], but we use the method of "interpolation determinant" instead of using an auxiliary function constructed from a "Siegel's Lemma". As in [8] or [6], we will first show (in Section 2) that it suffices to prove a weak form of Theorem 1. Then, in Section 3, we prove this weak version of Baker's theorem.

2. A weak version of Baker's theorem

In this section, we show that Theorem 1 follows from the following result:

Theorem 2. *Let $n \geq 1$ be an integer, let $\beta_1, \dots, \beta_n$ be n elements of $\bar{k}$ such that $1, \beta_1, \dots, \beta_n$ are k-linearly independent, and such that $|\beta_i| \leq 1$ for all i. Let further $\alpha_1, \dots, \alpha_{n+1}$ be $n+1$ elements of $D_\rho \cap \bar{k}$. Assume that the numbers $\operatorname{Log} \alpha_1, \dots, \operatorname{Log} \alpha_{n+1}$ are linearly independent over k. Then*

$$\operatorname{Log} \alpha_{n+1} - (\beta_1 \operatorname{Log} \alpha_1 + \cdots + \beta_n \operatorname{Log} \alpha_n) \neq 0.$$

Proof of Theorem 1, assuming Theorem 2. Suppose that Theorem 1 is false. Then there exist an integer $n \geq 1$ and elements $\alpha_1, \dots, \alpha_n \in \bar{k} \cap D_\rho$ such that $\operatorname{Log} \alpha_1, \dots, \operatorname{Log} \alpha_n$ are k-linearly independent but $\bar{k}$-linearly dependent. Choose n minimal satisfying this condition. We note that $n \geq 2$. Let $\beta_1, \dots, \beta_n \in \bar{k}$, not all zero, such that

$$\sum_{i=1}^{n} \beta_i \operatorname{Log} \alpha_i = 0. \tag{3}$$

We claim that the minimality of n implies that $\beta_1, \dots, \beta_n$ are k-linearly independent. Indeed, suppose that they are not. Then, by renumbering if necessary, we have a relation

$$a_n \beta_n = \sum_{i=1}^{n-1} a_i \beta_i \tag{4}$$

with $a_i \in A$ $(1 \leq i \leq n)$ and $a_n \neq 0$. Then (3) yields

$$\sum_{i=1}^{n-1} (a_i \operatorname{Log} \alpha_n + a_n \operatorname{Log} \alpha_i) \beta_i = 0. \tag{5}$$

[1]The nonhomogeneous version of Baker's theorem would state that under the assumptions of Theorem 1, the $n+1$ numbers $1, \operatorname{Log} \alpha_1, \dots, \operatorname{Log} \alpha_n$ are linearly independent over $\bar{k}$.

Now, by the functional equation (2), we have

$$a_i \operatorname{Log} \alpha_n + a_n \operatorname{Log} \alpha_i = \operatorname{Log}(\Phi_{a_i}(\alpha_n) + \Phi_{a_n}(\alpha_i)) = \operatorname{Log} \alpha_i'$$

with $\alpha_i' = \Phi_{a_i}(\alpha_n) + \Phi_{a_n}(\alpha_i) \in \bar{k} \cap D_\rho$. Moreover, $\operatorname{Log} \alpha_1', \dots, \operatorname{Log} \alpha_{n-1}'$ are k-linearly independent. So these elements are $\bar{k}$-linearly independent by minimality of n. The relation (5) then implies $\beta_1 = \cdots = \beta_{n-1} = 0$, hence $\beta_n = 0$ by (4), which is a contradiction. Hence $\beta_1, \dots, \beta_n$ are k-linearly independent as claimed. Applying now Theorem 2 to the numbers $\operatorname{Log} \alpha_1, \dots, \operatorname{Log} \alpha_n$, we see that (3) cannot hold. This is a contradiction. □

3. Proof of Theorem 2

This section is devoted to the proof of Theorem 2. The argument follows the same lines as a standard transcendence proof. We assume that the conclusion of Theorem 2 is false and we try to derive a contradiction. For this, we construct first (in Sections 3.1 and 3.2) a certain non-zero algebraic number Δ. This number is defined as a minor of order L of a certain matrix having L rows. The fact that such a minor exists (*i.e.* is not zero) rests on the use of a zero estimate due to Denis. In a second step (Section 3.3), we find an upper bound for $|\Delta|$ using analytic arguments (namely, a Schwarz Lemma). In a third step (Section 3.4), we find a lower bound for $|\Delta|$, using the fact that Δ is algebraic and non zero: we use here the so-called Liouville's inequality. The upper bound and the lower bound being contradictory, we get the desired contradiction.

From now on, we suppose that we are given elements $\alpha_1, \dots, \alpha_{n+1}$, $\beta_1, \dots, \beta_n$ in $\bar{k}$ as in Theorem 2. For $1 \leq i \leq n+1$ we define $\lambda_i := \operatorname{Log} \alpha_i$ and we suppose that the following equality holds:

$$\lambda_{n+1} = \beta_1 \lambda_1 + \cdots + \beta_n \lambda_n. \tag{6}$$

We denote by $c_0, c_1, \dots, c_4$ fixed real numbers depending only on q, n and $\alpha_1, \dots, \alpha_{n+1}$, $\beta_1, \dots, \beta_n$ (such numbers will be called "constants"). These numbers will appear during the proof and could be made explicit, but we did not carry out this task.

3.1. **Construction of a matrix $\mathcal{M}$.**

We begin by defining positive integers T_1, T_2, S as follows: S is a constant chosen sufficiently large, and

$$T_1 = \lfloor q^{(1+1/n)S} / S^3 \rfloor, \qquad T_2 = S^{2n}. \tag{7}$$

Here, "sufficiently large" means that all the inequalities that will occur in the proof below are satisfied. In particular, we note that the choice of S depends on the constants $c_0, \dots, c_4$ defined before. As for the choice (7), it is imposed by the various constraints on S, T_1, T_2 that will appear during the proof (see Remark 1).

We introduce the following notation. For any $\boldsymbol{s} = (s_1, \dots, s_{n+1}) \in A^{n+1}$, we define

$$\deg \boldsymbol{s} := \max_{1 \leq i \leq n+1} \{\deg s_i\}$$

and we define the two sets

$$\mathcal{T} = \{(\tau_1, \dots, \tau_n, t) \in \mathbf{N}^n \times \mathbf{N} \mid \tau_1 + \cdots + \tau_n \leq T_1, 0 \leq t \leq T_2\},$$

and

$$\mathcal{S} = \{\boldsymbol{s} \in A^{n+1} \mid \deg \boldsymbol{s} \leq S\}.$$

For every $(\boldsymbol{\tau}, t) \in \mathcal{T}$ with $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$, we consider the function $f_{(\boldsymbol{\tau},t)} : D_\rho^n \to \mathbf{C}_P$ defined by

$$f_{(\boldsymbol{\tau},t)}(z_1, \dots, z_n) = z_1^{\tau_1} \dots z_n^{\tau_n} \big(e(\sum_{i=1}^{n} \lambda_i z_i)\big)^t, \tag{8}$$

and for any $\boldsymbol{s} \in \mathcal{S}$ we define the algebraic points

$$\zeta_{\boldsymbol{s}} = (s_1 + s_{n+1}\beta_1, \dots, s_n + s_{n+1}\beta_n) \in \bar{k}^n. \tag{9}$$

Let L be the cardinal of $\mathcal{T}$. We have

$$L = \binom{T_1 + n}{n}(T_2 + 1). \tag{10}$$

We choose any ordering of the sets $\mathcal{T}$ and $\mathcal{S}$, and we consider the matrix

$$\mathcal{M} = \big(f_{(\boldsymbol{\tau},t)}(\zeta_{\boldsymbol{s}})\big)_{(\boldsymbol{\tau},t),\boldsymbol{s}}$$

where the rows are indexed by $(\boldsymbol{\tau}, t) \in \mathcal{T}$ and the columns are indexed by $\boldsymbol{s} \in \mathcal{S}$.

We note that the entries of $\mathcal{M}$ are actually elements of $\bar{k}$. Indeed, writing $\zeta_{\boldsymbol{s}} = (\zeta_{\boldsymbol{s}}^{(1)}, \dots, \zeta_{\boldsymbol{s}}^{(n)})$ and using the hypothesis (6), we have

$$\sum_{i=1}^{n} \lambda_i \zeta_{\boldsymbol{s}}^{(i)} = \sum_{i=1}^{n} \lambda_i s_i + (\sum_{i=1}^{n} \lambda_i \beta_i) s_{n+1} = \sum_{i=1}^{n+1} \lambda_i s_i,$$

hence

$$e\big(\sum_{i=1}^{n} \lambda_i \zeta_{\boldsymbol{s}}^{(i)}\big) = \sum_{i=1}^{n+1} \Phi_{s_i}(\alpha_i) \in \bar{k}$$

and thus

$$\mathcal{M} = \Big((s_1 + s_{n+1}\beta_1)^{\tau_1} \cdots (s_n + s_{n+1}\beta_n)^{\tau_n} \big(\sum_{i=1}^{n+1} \Phi_{s_i}(\alpha_i)\big)^t\Big)_{(\boldsymbol{\tau},t),\boldsymbol{s}} \tag{11}$$

has algebraic entries.

3.2. **Construction of a minor Δ.** Observe that by (7) and (10) we have $L \leq 2^{n+1} T_1^n T_2 < q^{(S+1)(n+1)} = \operatorname{card} \mathcal{S}$, so the rank of the matrix $\mathcal{M}$ is at most L. The aim of this section is to prove that this rank is exactly L.

Proposition 1. *The rank of the matrix $\mathcal{M}$ is equal to L.*

To prove this proposition we will use a zero estimate due to Denis [6]. To state his result, we need to introduce further notation. If $N \geq 1$ is an integer, let us denote by $\operatorname{End}_{\mathbf{F}_q-\mathrm{lin}}(\mathbf{G}_a^N)$ the $\mathbf{F}_q$-algebra of $\mathbf{F}_q$-linear endomorphisms of $\mathbf{G}_a^N$, and by $F : (x_1, \dots, x_N) \to (x_1^q, \dots, x_N^q)$ the Frobenius map on $\mathbf{G}_a^N$. Recall that a T-module of dimension N and rank $r \geq 0$ is a pair $G = (\mathbf{G}_a^N, \Psi)$, where $\Psi : A \to \operatorname{End}_{\mathbf{F}_q-\mathrm{lin}}(\mathbf{G}_a^N)$ is an injective homomorphism of $\mathbf{F}_q$-algebras such that $\Psi(T) = a_0 F^0 + \dots + a_r F^r$, where $a_i \in M_N(\mathbf{C}_P)$ $(0 \leq i \leq r)$, $a_r \neq 0$, and the only eigenvalue of a_0 is T. When $N = 1$ and $r = 0$, we get the "trivial" T-module, whose action on $\mathbf{G}_a$ is the usual scalar action.

A morphism $\varphi : G_1 \to G_2$ of T-modules is a morphism of algebraic groups that commutes with the actions of A. It is called an isogeny if it is surjective with finite kernel. We call sub-T-module of a T-module $(\mathbf{G}_a^N, \Psi)$ any connected algebraic subgroup H of $\mathbf{G}_a^N$ such that $\Psi_a(H) \subset H$ for all $a \in A$. If H is such a sub-T-module, we define $\deg H$ as the projective degree of its Zariski closure $\overline{H}$ in $\mathbf{P}^N$ via the embedding $\mathbf{G}_a^N \hookrightarrow \mathbf{P}^N$. A T-module $(\mathbf{G}_a^N, \Psi)$ is said to be simple if its only sub-T-modules are $\{0\}$ and $\mathbf{G}_a^N$.

If $G_1 = (\mathbf{G}_a^{n_1}, \Psi_1)$ and $G_2 = (\mathbf{G}_a^{n_2}, \Psi_2)$ are two T-modules, we will denote by $G_1 \times G_2$ the T-module $(\mathbf{G}_a^{n_1+n_2}, \Psi_1 \times \Psi_2)$, where $\Psi_1 \times \Psi_2$ is the diagonal action on $\mathbf{G}_a^{n_1} \times \mathbf{G}_a^{n_2}$. We define similarly a product of finitely many T-modules. In the zero estimate below, we will consider a product of T-modules $G = G_1^{n_1} \times \cdots \times G_m^{n_m}$, where $G_i = (\mathbf{G}_a^{d_i}, \Psi_i)$ is a T-module of dimension d_i. We will denote by $\mathbf{C}_P[X_{i,1}, \dots, X_{i,n_i d_i}]$ the coordinate ring of $G_i^{n_i}$, and by $X_i = (X_{i,1}, \dots, X_{i,n_i d_i})$ its set of variables. If $Q \in \mathbf{C}_P[X_1, \dots, X_m]$ is any polynomial, we will denote by $\deg_{X_i} Q$ its partial degree with respect to X_i. Finally, if $(\mathbf{G}_a^N, \Psi)$ is a T-module and $\Gamma = \{\gamma_1, \dots, \gamma_g\}$ is a finite set of points of $\mathbf{C}_P^N$, we denote, for any integer $S \geq 0$,

$$\Gamma(S) := \left\{ \sum_{i=1}^{g} \Psi_{s_i}(\gamma_i) \mid s_1, \dots, s_g \in A, \deg s_i \leq S \right\}.$$

We can now state the result of Denis.

Theorem 3 (Zero estimate)**.** *Let $G_i = (\mathbf{G}_a^{d_i}, \Psi_i)$ $(1 \leq i \leq m)$ be m T-modules of dimension d_i and rank $r_i \geq 0$. Suppose that G_i is simple for all i and that $G_1, \dots, G_m$ are pairwise non-isogeneous. Let $n_1, \dots, n_m$ be positive integers, and put $G = G_1^{n_1} \times \cdots \times G_m^{n_m}$, $N = d_1 n_1 + \cdots + d_m n_m$. Let further $S \geq 0$ be an integer and Γ a finite set of points in $\mathbf{C}_P^N$. Suppose that there exists a non-zero polynomial $Q \in \mathbf{C}_P[X_1, \dots, X_m]$ which vanishes on $\Gamma(S)$, and such that $\deg_{X_i} Q \leq L_i$ $(1 \leq i \leq m)$, where $L_i \in \mathbf{Z}_{>0}$. Then there exist sub-T-modules H_i of $G_i^{n_i}$ such that the product $H = H_1 \times \cdots \times H_m$ is distinct from $\mathbf{G}_a^N$ and such that*

$$\operatorname{card}\big((\Gamma(S-N+1)+H)/H\big) \cdot \frac{(\dim H)!}{(\dim H_1)! \dots (\dim H_m)!} \cdot \prod_{i=1}^{m} \deg H_i$$
$$\leq \frac{N!}{(n_1 d_1)! \dots (n_m d_m)!} \cdot \prod_{i=1}^{m} \deg G_i^{n_i} \cdot \prod_{i=1}^{m} (\kappa_i L_i)^{d_i n_i - \dim H_i},$$

where $\kappa_i = q^{r_i(N-1)}$.

Proof. See [6, Theorem 2]. □

Proof of Proposition 1. Suppose that $\operatorname{rank}(\mathcal{M}) < L$. Then there exists a non-trivial linear combination of the rows of $\mathcal{M}$ which vanishes, that is, by (11),

$$\sum_{(\boldsymbol{\tau},t)} \lambda_{(\boldsymbol{\tau},t)} (s_1 + s_{n+1}\beta_1)^{\tau_1} \dots (s_n + s_{n+1}\beta_n)^{\tau_n} \big(\sum_{i=1}^{n+1} \Phi_{s_i}(\alpha_i)\big)^t = 0$$

for all $\boldsymbol{s} \in \mathcal{S}$. Define the following points of $\overline{k}^{n+1}$:

$$\gamma_1 = (1, 0, \dots, 0, \alpha_1), \dots, \gamma_n = (0, \dots, 0, 1, \alpha_n), \gamma_{n+1} = (\beta_1, \dots, \beta_n, \alpha_{n+1}),$$

and consider the T-module $G = G_1^n \times G_2$, where G_1 is the trivial T-module of dimension 1 and where $G_2 = (\mathbf{G}_a, \Phi)$ is the Carlitz module. We note that G_1 and G_2 are simple and non-isogeneous. Moreover, if Ψ denotes the action of G, we have, for all $\boldsymbol{s} \in \mathcal{S}$,

$$\Psi_{s_1}(\gamma_1) + \cdots + \Psi_{s_{n+1}}(\gamma_{n+1}) = (s_1 + s_{n+1}\beta_1, \dots, s_n + s_{n+1}\beta_n, \sum_{i=1}^{n+1} \Phi_{s_i}(\alpha_i)).$$

Thus, we see that the non zero polynomial

$$Q(X_1,\dots,X_n,Y)=\sum_{(\boldsymbol{\tau},t)}\lambda_{(\boldsymbol{\tau},t)}X_1^{\tau_1}\dots X_n^{\tau_n}Y^t$$

vanishes on $\Gamma(S)$. Moreover, $\deg_X\leq T_1$ and $\deg_Y Q\leq T_2$. Applying Theorem 3, we find that there is a sub-$\mathbf{C}_P$-vector space $H_1\subset\mathbf{C}_P^n$ and a connected algebraic subgroup $H_2\subset\mathbf{G}_a$ such that $H=H_1\times H_2\neq\mathbf{G}_a^{n+1}$, and

$$\operatorname{card}\big((\Gamma(S-n)+H)/H\big)\leq c_0T_1^{n-\dim H_1}T_2^{1-\dim H_2}. \tag{12}$$

Suppose first that $H_2=\{0\}$. Then clearly, by considering the last coordinate only,

$$\operatorname{card}\big((\Gamma(S-n)+H)/H\big)\geq\operatorname{card}\{\sum_{i=1}^{n+1}\Phi_{s_i}(\alpha_i)\mid\deg s_i\leq S-n\}. \tag{13}$$

Since $\sum_{i=1}^{n+1}\Phi_{s_i}(\alpha_i)=e(\sum_{i=1}^{n+1}s_i\lambda_i)$ and $e:D_\rho\to D_\rho$ is injective, the k-linear independence of $\lambda_1,\dots,\lambda_{n+1}$ implies that the points $\sum_{i=1}^{n+1}\Phi_{s_i}(\alpha_i)$, $\deg s_i\leq S-n$, are all distincts. Hence (12) and (13) yield

$$q^{(S-n+1)(n+1)}\leq c_0T_1^nT_2,$$

which is a contradiction by the choice of the parameters (7) (recall that S is chosen sufficiently large).

Suppose now that $H_2=\mathbf{G}_a$. Since $H=H_1\times H_2\neq\mathbf{G}_a^{n+1}$, we have $\dim H_1\leq n-1$. Let us define

$$\begin{aligned}\Delta[S-n] &:= \{(s_1+s_{n+1}\beta_1,\dots,s_n+s_{n+1}\beta_n)\mid\deg s_i\leq S-n\}\\ &= \{\sum_{i=1}^{n+1}s_i\xi_i\mid\deg s_i\leq S-n\},\end{aligned}$$

where $(\xi_1,\dots,\xi_n)$ is the canonical basis of $\mathbf{C}_P^n$ and $\xi_{n+1}=(\beta_1,\dots,\beta_n)$. We have

$$\operatorname{card}\big((\Gamma(S-n)+H)/H\big)=\operatorname{card}\big((\Delta[S-n]+H_1)/H_1\big).$$

Put $r=\dim(\mathbf{C}_p^n/H_1)$ and $\rho=\dim_k(<\xi_1,\dots,\xi_{n+1}>_k+H_1/H_1)$, where $<\xi_1,\dots,\xi_{n+1}>_k$ denotes the sub-k-vector space of $\mathbf{C}_P^n$ spanned by $\xi_1,\dots,\xi_{n+1}$. We claim that $\rho\geq r+1$. Indeed, there are already at least r elements among $\xi_1,\dots,\xi_n$ that are k-linearly independent modulo H_1, say $\xi_{i_1},\dots,\xi_{i_r}$. Hence $\rho\geq r$. Suppose that we have $\rho=r$. Then $\dim_k(<\xi_1,\dots,\xi_n>_k+H_1/H_1)=r$ and the vector space $\mathbf{C}_P^n/H_1$ is defined over k, hence H_1 is also defined over k. Now, $\xi_{i_1},\dots,\xi_{i_r},\xi_{n+1}$ are k-linearly dependent modulo H_1, so we can write $\xi_{n+1}=\sum_{\ell=1}^r\mu_{i_\ell}\xi_{i_\ell}+h$, where $\mu_{i_\ell}\in k$ and $h\in H_1$. Thus we have $(\beta_1-\mu_1,\dots,\beta_n-\mu_n)\in H_1$, where we have put $\mu_i=0$ if $i\neq\{i_1,\dots,i_r\}$. But then the point $(\beta_1-\mu_1,\dots,\beta_n-\mu_n)$ is contained in a hyperplane of $\mathbf{C}_P^n$ defined over k, which contradicts the k-linear independence of $1,\beta_1,\dots,\beta_n$. Hence $\rho\geq r+1$, as claimed. It readily follows from this that

$$\operatorname{card}\big((\Gamma(S-n)+H)/H\big)=\operatorname{card}\big((\Delta[S-n]+H_1)/H_1\big)\geq q^{(S-n+1)(r+1)},$$

hence (12) yields

$$q^{S-n+1}\leq c_0(T_1/q^{S-n+1})^r.$$

Since $T_1\geq q^{S-n+1}$, we get $q^{S-n+1}\leq c_0(T_1/q^{S-n+1})^n$, which again contradicts the choice of the parameters (7). $\square$

It follows from Proposition 1 that there is a $L\times L$ minor of $\mathcal{M}$ which is not zero. We choose L column indices $\boldsymbol{s}_1,\dots,\boldsymbol{s}_L$ of $\mathcal{S}$ such that the corresponding minor does not vanish. For ease of notation, in the sequel we rename the points $\zeta_{\boldsymbol{s}_1},\dots,\zeta_{\boldsymbol{s}_L}$ as $\zeta_1,\dots,\zeta_L$, and similarly, we rename the functions $\{f_{(\boldsymbol{\tau},t)}\mid(\boldsymbol{\tau},t)\in\mathcal{T}\}$ as f_λ, $1\le\lambda\le L$. We set

$$\Delta:=\det(f_\lambda(\zeta_\mu))_{1\le\lambda,\mu\le L}.$$

3.3. **Upper bound for $|\Delta|$.** In this section we will prove:

Proposition 2. *Let $E_0=\rho/\max_{1\le i\le n}\{|\lambda_i|\}$. Then we have*

$$\log_q|\Delta|\le-\frac{n}{2e}L^{1+1/n}\log_q E_0.$$

We will need the ultrametric version of the Schwarz lemma. If $R>0$ is a real number, we define $D(0,R)=\{z\in\mathbf{C}_P\mid|z|<R\}$. We will say that a function $\psi:D(0,R)\to\mathbf{C}_P$ is analytic in $D(0,R)$ if we can write $\psi(z)=\sum_{n\ge0}a_nz^n$ for all $z\in D(0,R)$. In that case, if r is any real such that $0<r<R$, we define

$$|\psi|_r:=\sup\{|\psi(z)|\mid|z|\le r\}.$$

Lemma 1 (Schwarz Lemma)**.** *Let $0<r\le R$ be two positive real numbers in the group of values $|\mathbf{C}_P^\times|=q^{\mathbf{Q}}$, and let ψ be a non zero analytic function in a disk containing strictly $D(0,R)$. If $M=\operatorname{ord}_{z=0}\psi(z)$, then*

$$|\psi|_r\le\left(\frac{r}{R}\right)^M|\psi|_R.$$

Proof. Define $\varphi(z)=z^{-M}\psi(z)$. By the maximum modulus principle, we have $|\varphi|_r=r^{-M}|\psi|_r$ and $|\varphi|_R=R^{-M}|\psi|_R$. The lemma now follows from the obvious inequality $|\varphi|_r\le|\varphi|_R$. □

Proof of Proposition 2. We introduce the following analytic function in one variable, for $|z|$ small :

$$D(z)=\det(f_\lambda(z\zeta_\mu))_{1\le\lambda,\mu\le L}.$$

We claim that this function is actually analytic in the disk $D(0,E_0)$. Indeed, if $\zeta_\mu=(\zeta_{\mu,1},\dots,\zeta_{\mu,n})$, one readily checks from the definition of the points ζ_μ (see (9)) and from the fact that $|\beta_i|\le1$ for all i, that $|\zeta_{\mu,i}|\le1$ for all i, hence $|\sum_{i=1}^n\lambda_i\zeta_{\mu,i}|\le\max_i|\lambda_i|$ and thus $z(\sum_{i=1}^n\lambda_i\zeta_{\mu,i})\in D_\rho$ if $|z|<E_0$. Coming back to the definition of the functions f_λ (see (8)), this proves the claim. We have moreover, if λ corresponds to the $n+1$-tuple $(\tau_1,\dots,\tau_n,t)\in\mathcal{T}$:

$$|f_\lambda(z\zeta_\mu)|\le|z|^{\tau_1+\cdots+\tau_n}\rho^t\quad\text{for all }z\in D(0,E_0).\tag{14}$$

We apply now the Schwarz Lemma to the function D with $r=1$ and $R\in q^{\mathbf{Q}}$ such that $r\le R<E_0$. We obtain

$$|\Delta|=|D(1)|\le R^{-M}|D|_R,$$

where $M=\operatorname{ord}_{z=0}D(z)$. We deduce from (14) the estimate $|D|_R\le R^{LT_1}$, hence $|\Delta|\le R^{-M+LT_1}$.

Let us estimate from below the multiplicity M at 0 of the function $D(z)$. We follow here almost verbatim [8], Lemmas 6.4 and 6.5. By multilinearity of the determinant and by expanding each function f_λ at $(0,\dots,0)$ as $f_\lambda(\boldsymbol{z})=\sum_{\boldsymbol{i}}f_{\lambda,\boldsymbol{i}}\boldsymbol{z}^{\boldsymbol{i}}$ (where $\boldsymbol{z}^{\boldsymbol{i}}$ means as usual $z_1^{i_1}\dots z_n^{i_n}$ when $\boldsymbol{z}=(z_1,\dots,z_n)$ and $\boldsymbol{i}=(i_1,\dots,i_n)$), we see that we may assume that each entry of $D(z)$ is a monomial of the form $z^{\|\boldsymbol{i}\|}\zeta_{\boldsymbol{\mu}}^{\boldsymbol{i}}$, where $\|\boldsymbol{i}\| := i_1 + \cdots + i_n$. In that case, we have the commun factor $z^{\|\boldsymbol{i}\|}$ in each row indexed by $\boldsymbol{i}$. Moreover, we may assume that two different rows correspond to two different indices $\boldsymbol{i}$, because otherwise the two rows would be identical and the corresponding determinant would be zero. We deduce from this that the vanishing order of $D(z)$ at 0 is at least equal to

$$\Theta_n(L) := \min\{\|\boldsymbol{i_1}\| + \cdots + \|\boldsymbol{i_L}\|\},$$

where the minimum runs over all the L-tuples $(\boldsymbol{i_1}, \ldots, \boldsymbol{i_L}) \in \mathbf{N}^n \times \cdots \times \mathbf{N}^n$ which are pairwise distinct. Lemma 6.5 of [8] yields the estimate $\Theta_n(L) > (n/e)L^{1+1/n}$ as soon as $L \geq (4n)^{2n}$. By the choice (7), this latter condition is satisfied. Summing up, we have obtained

$$\log_q |\Delta| \leq (LT_1 - \frac{n}{e}L^{1+1/n}) \log_q R \leq -\frac{n}{2e}L^{1+1/n} \log_q R$$

(since $T_1 \leq (n/2e)L^{1/n}$ by (7)). Letting now R tend to E_0, we obtain the proposition. □

3.4. **Lower bound for $|\Delta|$ and conclusion.** In this section we prove the following lower bound for $|\Delta|$, from which we derive the desired contradiction.

Proposition 3. *The following inequality holds:*

$$\log_q |\Delta| \geq -c_2 LS(T_1 + T_2 q^S).$$

To prove this proposition we will use the "Liouville's inequality". We will need the notion of height of an algebraic point of $\mathbf{P}^N(\bar{k})$. We recall for convenience the definition and the basic properties we will use.

If $(\xi_0 : \xi_1 : \cdots : \xi_N)$ is a point of $\mathbf{P}^N(\bar{k})$, we define its height by the formula

$$h(\xi_0 : \xi_1 : \cdots : \xi_N) = \frac{1}{[K:k]} \sum_{w \in M_K} d_w \max\{-w(\xi_0), \ldots, -w(\xi_N)\}, \tag{15}$$

where K/k is any finite extension such that $\xi_0, \ldots, \xi_N \in K$, M_K is the set of non trivial places of K, d_w is the degree over $\mathbf{F}_q$ of the residue class field at w, and the valuation w is normalized so that $w(K^\times) = \mathbf{Z}$. The properties of the valuations show that this definition does not depend on the choice of K containing $\xi_0, \ldots, \xi_N$, and the product formula shows that it is independent of the chosen projective coordinates $(\xi_0, \ldots, \xi_N)$ for the point. If ξ is an element of $\bar{k}$, we define $h(\xi)$ by $h(\xi) := h(1 : \xi)$.

Lemma 2. *Let $f \in A[X_1, \ldots, X_{n+1}, Y_1, \ldots, Y_n]$ be any non zero polynomial. We have*

$$h(f(\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \ldots, \beta_n)) \leq c_1 \deg f + \delta(f),$$

where $c_1 = h(1 : \alpha_1 : \cdots : \alpha_{n+1} : \beta_1 : \cdots : \beta_n)$ and where $\delta(f)$ denotes the maximum of the degrees (in T) of the coefficients of f.

Proof. Let $K = k(\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \ldots, \beta_n)$. One easily checks that for any place w of K, one has

$$\max\{0, -w(f(\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \ldots, \beta_n))\}$$
$$\leq \deg f \, \max\{0, -w(\alpha_1), \ldots, -w(\alpha_{n+1}), -w(\beta_1), \ldots, -w(\beta_n)\} + c_w$$

with $c_w = 0$ if $w \nmid \infty$ and $c_w = e_w\delta(f)$ if $w \mid \infty$ (here e_w is the ramification index at w). The lemma follows from this and the definition of the height (15). □

Corollary 1 (Liouville's inequality). *With the notations of Lemma 2, we have, if* $f(\alpha_1,\dots,\alpha_{n+1},\beta_1,\dots,\beta_n) \neq 0$,

$$\log_q |f(\alpha_1,\dots,\alpha_{n+1},\beta_1,\dots,\beta_n)| \geq -[K:k](c_1 \deg f + \delta(f)),$$

where $K = k(\alpha_1,\dots,\alpha_{n+1},\beta_1,\dots,\beta_n)$.

Proof. Put $\xi := f(\alpha_1,\dots,\alpha_{n+1},\beta_1,\dots,\beta_n)$. Since $(1:\xi) = (\xi^{-1}:1)$, we have $h(\xi) = h(\xi^{-1})$. Hence

$$\log_q |\xi^{-1}| = -v(\xi^{-1}) \leq \max\{0, -v(\xi^{-1})\} \leq [K:k]h(\xi^{-1}) = [K:k]h(\xi).$$

Now, Lemma 2 yields the result. □

Proof of Proposition 3. From the definition of Δ and the expression (11), we see that we can write $\Delta = f(\alpha_1,\dots,\alpha_{n+1},\beta_1,\dots,\beta_n)$, where f is a polynomial of $A[X_1,\dots,X_{n+1},Y_1,\dots,Y_n]$ of the form

$$f(X_1,\dots,X_{n+1},Y_1,\dots,Y_n) = \det\Big((s_1+s_{n+1}Y_1)^{\tau_1}\dots(s_n+s_{n+1}Y_n)^{\tau_n}\Big(\sum_{i=1}^{n+1}\Phi_{s_i}(X_i)\Big)^t\Big)_{(\boldsymbol{\tau},t),\boldsymbol{s}}$$

($(\boldsymbol{\tau},t)$ runs over all the elements of $\mathcal{T}$ and $\boldsymbol{s}$ runs over a subset of $\mathcal{S}$ of cardinality L). We have, since $\deg(\Phi_{s_i}(X_i)) = q^{\deg s_i} \leq q^S$:

$$\deg_X f \leq LT_2q^S \quad \text{and} \quad \deg_Y f \leq LT_1.$$

Moreover, the coefficients of each polynomial $\Phi_{s_i}(X_i)$ are elements of A of degree in T at most $q^{\deg s_i}\deg s_i \leq Sq^S$, hence the coefficients of f have a degree in T at most $L(T_1S+T_2Sq^S)$. It follows from these estimates and from Liouville's inequality that

$$\log_q |\Delta| \geq -c_2LS(T_1+T_2q^S).$$

□

End of the proof of Theorem 2. By Propositions 2 and 3, we have:

$$\frac{n}{2e}L^{1+1/n}\log_q E_0 \leq c_2LS(T_1+T_2q^S)$$

or

$$L^{1/n} \leq c_3S(T_1+T_2q^S).$$

Since by (10) we have $L \geq T_1^nT_2/n!$, we obtain

$$T_1T_2^{1/n} \leq c_4(T_1S+T_2Sq^S).$$

But this inequality contradicts the choice of the parameters (7). Thus the assumption (6) was false, which completes the proof of Theorem 2. □

Remark 1. If we keep track of all the inequalities that the parameters S, T_1 and T_2 should satisfy in order that the above proof works, we see that these parameters have to be sufficiently large and should satisfy the following three conditions: (i) $T_1 \geq q^{S-n+1}$, (ii) $c_0T_1^nT_2 < q^{(S-n+1)(n+1)}$ and (iii) $c_4(T_1S+T_2Sq^S) < T_1T_2^{1/n}$. The definition (7) has been chosen to fulfill these conditions.

References

[1] A. Baker, *Linear forms in the logarithms of algebraic numbers I, II, III*, Mathematika 13 (1966), 204-216; ibid. 14 (1967), 102-107; ibid. 14 (1967), 220–228.

[2] V. Bosser, *Minorations de formes linéaires de logarithmes pour les modules de Drinfeld*, J. Number Theory 75 (1999), 279–323.

[3] A. Brumer. *On the units of algebraic number fields*, Mathematika 14 (1967), 121–124.

[4] S. David and N. Hirata-Kohno. *Linear forms in elliptic logarithms*, J. Reine Angew. Math. 628 (2009), 37–89.

[5] L. Denis. *Dérivées d'un module de Drinfeld et transcendance*, Duke Math. J. 80 (1995), 1–13.

[6] L. Denis. *Théorème de Baker et modules de Drinfeld*, J. Number Theory 43 (1993), 203–215.

[7] E. Gaudron. *Mesures d'indépendance linéaire de logarithmes dans un groupe algébrique commutatif*, Invent. Math. 162, n.1 (2005), 137–188.

[8] M Waldschmidt. *Diophantine approximation on linear algebraic groups*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag (2000).

[9] G. Wüstholz. *A panorama of number theory or the view from Baker's garden*, Papers from the Conference on Number Theory and Diophantine Geometry at the Gateway to the Millennium held in Zürich, August 28–September 3, 1999. Cambridge University Press (2002).

[10] J. Yu. *Transcendence and Drinfeld modules: several variables*, Duke Math. J. 58, n.3 (1989), 559–575.

[11] J. Yu. *Analytic homomorphisms into Drinfeld modules*, Ann. of Math. 145, n.2 (1997), 215–233.

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